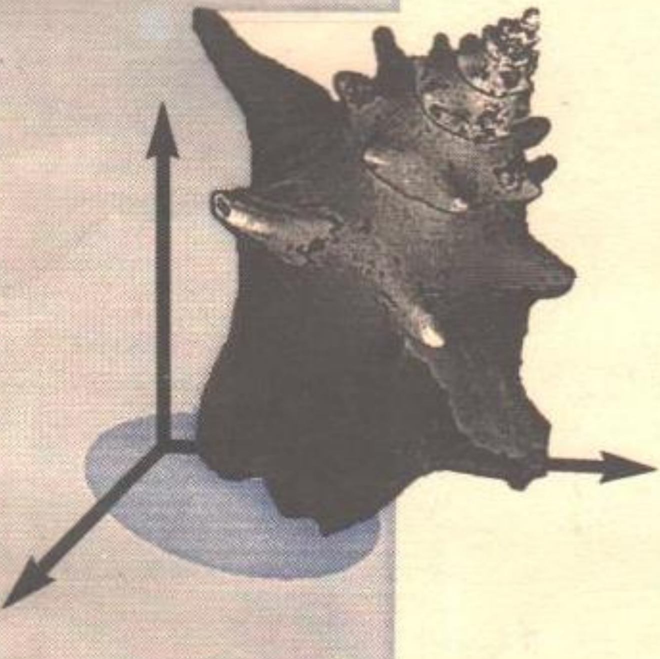
АЛГЕБРА И ТЕОРИЯ МОДЕЛЕЙ



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Algebra and Model Theory

Collection of papers
edited by A.G.Pinus and K.N.Ponomaryov

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The papers of this book are devoted to some problems of algebra and model theory.

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Preface

There are two different ways of approaching the investigation of an algebraic system. One way lead us to universal algebra. The other one to model theory. So these fields of math are mutually enriching areas of research.

In Russia an idea for organization a joint conference for mathematicians in these fields of research belongs to prof.E.A.Palutin and prof. A.G.Pinus. In 1991 they have organized a conference on algebra and model theory in Saratov. It formed a part of Suslin conference.

Afterwards a camping center "Erlagol" of Novosibirsk State Technical University was chosen as a central place for the following schools of algebra and model theory. First international school "Intermediate problems of universal algebra and model theory" was held in 1995. The second conference had place in 1997.

Organization committee of a conference thanks Russian Fund of Fundamental Research for supporting the organization of both conferences. Grant N 97-01-10037.

Needless to say that mathematicians from different branches of math were participants of these conferences. But the main topic of all reports was in the circle of problems of model theory and universal algebra. We'll produce below lists of organization committees of these conferencies, programs and some papers of participants.

A.G.Pinus, K.N.Ponomaryov

SCHOOL ON UNIVERSAL ALGEBRA AND MODEL THEORY. (23-27 OF SEPTEMBER 1991, SARATOV)

E.A.Palutin (Novosibirsk). *Stable theories.*

O.V.Belegradek (Kemerovo). *Stable theories in algebra.*

A.G.Pinus (Novosibirsk). *Skeletons of varieties.*

M.V.Wolkov (Ekaterinburg). *Lattices of semigroup varieties and of universal algebras.*

A.I.Omarov (Alma-Aty). *P-formulae and algebraic systems with a normal evaluation.*

A.I.Malcev (Novosibirsk). *Post varieties.*

R.A.Bairamov (Baku). *Between clones and lattices of varieties.*

N.A.Shchuchkin (Volgograd). *Finite conditions for nilpotent algebras.*

K.K.Kaarly (Tartu). *On arithmetically affin-complete varieties.*

D.A.Bredihin (Saratov). *On relation algebras.*

E.Baisalov (Alma-Aty). *Countable models of stable theories.*

K.Shagirov (Alma-Aty). *Stable geometric lattices.*

B.M.Vernikov (Ekaterinburg). *Completion conditions in lattices of varieties.*

T.A.Ershova (Ekaterinburg). *Lattices of varieties of semigroups with a completely regular square.*

INTERMEDIATE PROBLEMS OF MODEL THEORY AND UNIVERSAL ALGEBRA (23 -27 OF JUNE 1995, ERLAGOL)

Organization committee: acad. Yu.L.Ershov (chairman), prof. E.A.Palutin, prof. A.G.Pinus, prof. B.Poizat

secretary: doc. S.V.Sudoplatov

23 of June. Chairman A.G.Pinus.

R.Willard (Waterloo, Canada). *Structured discriminatory varieties (an introduction).*

O.V.Belegradek (Kemerovo, Russia). *Group-like quasivarieties.*

F.Wagner (Oxford, England). *Nilpotency in MG-groups.*

Chairman B.I.Zilber.

B.Poizat (Lyon, France). *Les petits cailloux. 1.*

R.El Bashir (Prague, Chech Republic). *Slender modules.*

A.Stepanova (Vladivostok, Russia). *On model completeness of regular S-acts.*

24 of June. Chairman O.V.Belegradek.

B.I.Zilber (Kemerovo, Russia). *Algebraic and analytic geometry via model theory. 1.*

R.Willard (Waterloo, Canada). *Structured discriminator varieties.*

B.Poizat (Luon, France). *Les petits cailloux. 2.*

Chairman L.Nevelski.

F.Wagner (Oxford, Great Britan). *Nilpotency in substable groups.*

D.Vasilyev (Novosibirsk, Rossia). *Categoricity problem and n-gedeneracy of an algebraic closure.*

Ja.L.Mordvinov. (Nojabrsk, Rossia). *On the skeletons of congruence-modular varieties.*

25 of June. Chairman F.Wagner.

A.Ivanov (Wroclaw, Poland). *Finite covers of ω-categorical structures.*

B.I.Zilber (Kemerovo, Rossia). *Algebraic and analytic georemety via model theory. 2.*

B.Seselia (Novi Sad, Yugoslavia). *Algebras with UCEP.*

Chairman B.Seselia.

L.Nevelski (Wroclaw, Poland). *M-rank in small superstable theories.*

A.Tepavcevic (Novi Sad, Yugoslavia). *On congruence intersection property.*

I.Bescennyi (Omsk, Rossia). *On quasi-identities of finite algebras.*

A.A.Vikentiev (Novosibirsk, Rossia). *Semantic distance between formulas.*

26 of June. Chairman E.Sukhanov.

L.Nevelski (Wroclaw, Poland). *M-rank in small superstable theories.*

A.G.Pinus (Novosibirsk, Russia). *Conditional terms, conditional varietes, conditional rational equivalence relation.*

K.N.Ponomaryov (Novosibirsk, Russia). *Covers of algebraic varieties and elementary theory of local fields.*

Chairman V.Roman'kov.

E.Sukhanov (Ekaterinburg, Russia). *Equational languages.*

B.Seselia (Novi Sad, Yugoslavia). *Lattice clasification of finite lattices by meet irrducibles.*

S.Sudoplatov (Novosibirsk, Russia).*Partial algebras associated with trigonometries of groups.*

A.Ivanov (Wroclaw, Poland).*Finite covers of w-categorical structures.*

27 of June. Chairman B.Poizat.

K.N.Ponomaryov (Novosibirsk, Russia).*Covers of algebraic varieties and elementary theory of local fields.*

A.G.Pinus (Novosibirsk, Russia).*Conditional terms, conditional varietes, conditional rational equivalence relation.*

V.Roman'kov (Omsk, Russia).*Universal theories and equations in solvable groups; up to date situation and possible development.*

Chairman K.N.Ponomaryov.

S.Lenjuk (Barnaul, Russia).*On the lattice of quasivarieties of metabelian groups.*

A.Zenkov (Barnaul, Russia).*On groups with infinite set of right orders.*

E.Ovchinnikova (Novosibirsk, Russia).*On regular s-acts.*

INTERMEDIATE PROBLEMS OF MODEL THEORY AND UNIVERSAL ALGEBRA
(17 -21 of June 1997, Erlagol)

Organization committee: prof. A.G.Pinus (chairman), acad. Yu.L.Ershov, prof. V.D.Mazurov, prof. E.A.Palutin

secretary: doc. K.N.Ponomaryov

17 of June. Chairman A.G.Pinus.

H.Rose (Capetown, Suide-Afrika). *On axiomatizability of some amalgamation classes of lattices.*

G.A.Noskov (Omsk, Russia). *Automatic groups.*

O.V.Belegradek (Kemerovo, Russia). *Quasi o-minimal theories.*

18 of June. Chairman H.Rose.

V.D.Mazurov (Novosibirsk, Russia). *On definability of finite groups by sets of orders of their elements.*

B.S.Baizhanov (Alma-Aty, Kazahstan). *Enrichments of models of weak o-minimal theories.*

C.Wolf (Darmshtadt, Germany). *From many-sorted to one-sorted structures.*

Chairman O.V.Belegradek

D.E.Palchunov (Novosibirsk, Russia). *Countable homogeneous I-algebras.*

S.I.Mardaev (Novosibirsk, Russia). *Stable points in Kripke models.*

A.V.Kravchenko (Novosibirsk, Russia). *Color families of graphs.*

19 of June. Chairman V.D.Mazurov.

K.N.Ponomaryov (Novosibirsk, Russia). *Semialgebraic sets.*

V.M.Kopytov (Novosibirsk, Russia). *Semilinear ordered groups.*

S.V.Sudoplatov (Novosibirsk, Russia). *On classification of group poligonometries.*

Chairman B.S.Baizhanov.

S.K.Mynbaeva (Alma-Aty, Kazakhstan). *Some questions of weakly o-minimal binary theories.*

M.S.Sheremet (Novosibirsk, Russia). *Difficulty of lattices of quasivarieties.*

S.V.Lenuk (Barnaul, Russia). *Filters in lattices of group quasivarieties.*

A.M.Popova (Novosibirsk, Russia). *Multiplicative structure of finitely generated rings.*

21 of June. Chairman V.M.Kopytov.

V.N.Glazkov (Novosibirsk, Russia). *Convergence structures, questions of continuation of homomorphisms of ordered groups and boolean algebras.*

A.G.Pinus (Novosibirsk, Russia). *Conditional terms and conditional varieties.*

B.Sh.Kulpeshov (Alma-Aty, Kazakhstan). *Weakly o-minimality of linear orders.*

О СВОЙСТВЕ МОНОТОННОСТИ СЛАБО О-МИНИМАЛЬНЫХ МОДЕЛЕЙ[1]

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Следующие определения 1-3 являются переформулировками определений, данных Д. Макферсоном, Д. Маркером, Ч. Стейнхорном в [1].

Определение 1 Пусть M — линейно упорядоченная модель сигнатуры σ, $\varphi(\bar{x}, x)$ — формула сигнатуры σ с $l(\bar{x}) = n$. Определим отношение эквивалентности на M^n:

$$\bar{a} \sim \bar{b} \overset{def}{\Longleftrightarrow} (\exists x_0)(\forall x > x_0)\varphi(\bar{a}, x) \leftrightarrow \varphi(\bar{b}, x).$$

(Мы можем определить подобное отношение эквивалентности по "нижней границе"формулы φ вместо "верхней"как выше.)

Пусть X — множество классов эквивалентности по $\sim$. Определим линейный порядок на $M \cup X$. Пусть $\bar{a}/_\sim \in X, c \in M$, полагаем $\bar{a}/_\sim < c$ в том и только в том случае, когда $M \models (\forall y)\varphi(\bar{a}, y) \to y < c$. Очевидно, что $\bar{a} \not\sim \bar{b}$ влечет $\bar{a}/_\sim < c < \bar{b}/_\sim$ или $\bar{b}/_\sim < c < \bar{a}/_\sim$ для некоторого $c \in M$. Назовем это множество X *сортом* в $\overline{M}$, где через $\overline{M}$ обозначается объединение всех сортов.

Определение 2 Пусть f — функция из упорядоченного множества I в упорядоченное множество K. Мы говорим, что:

(i) f *локально возрастает* на I, если для любого $a \in I$ существует бесконечный интервал $J \subseteq I$ такой, что $a \in J$ и $f|J$ — возрастающая функция.

(ii) f *локально убывает* на I, если для любого $a \in I$ существует бесконечный интервал $J \subseteq I$ такой, что $a \in J$ и $f|J$ — убывающая функция.

[1] Автор премного благодарен Виктору Вербовскому за ценные замечания при обсуждении результатов работы.

(iii) f *локально постоянна* на I, если для любого $a \in I$ существует бесконечный интервал $J \subseteq I$ такой, что $a \in J$ и $f|J$ — константная функция.

(iv) f *имеет локальный минимум в каждой точке* I, если для любого $a \in I$ существует бесконечный интервал $J \subseteq I$ такой, что a — точка минимума $f|J$.

Функции, имеющие локальный максимум в каждой точке некоторого множества, определяются аналогично пункту (iv) определения 2.

Определение 3

- Пусть f — частичная функция из упорядоченного множества I в упорядоченное множество K. Будем говорить, что функция f имеет *свойство монотонности*, если существует разбиение $\mathrm{dom}(f)$ на множества $X, I_0, \ldots, I_{n-1}$ такое, что X конечно, $I_i, i < n$ бесконечные выпуклые, и для каждого $i < n$ $f|I_i$ удовлетворяет одному из пунктов (i)–(iii) определения 2.
- Пусть M — линейно упорядоченная модель, будем говорить, что M имеет свойство монотонности, если любая определимая в M частичная 1-местная функция (как в саму модель, так и в некоторый сорт) имеет свойство монотонность.

Теорема *Любая слабо о-минимальная модель имеет свойство монотонности.*

Первые шаги нашего доказательства, включая утверждение 1, аналогичны доказательству теоремы о свойстве монотонности для моделей слабо о-минимальных теорий в [1], но для полноты и удобства чтения мы приводим некоторые утверждения из [1] без доказательства.

Пусть M — слабо о-минимальная модель, $h : M \to A$ — определимая частичная функция в некоторый определимый сорт A. Определим следующие формулы:

$$\varphi_0(x) : \quad (\exists x_1 > x)(\forall y \in (x, x_1))h(y) < h(x),$$

$$\begin{aligned}
\varphi_1(x):&\quad (\exists x_1 > x)(\forall y \in (x, x_1))h(y) = h(x),\\
\varphi_2(x):&\quad (\exists x_1 > x)(\forall y \in (x, x_1))h(y) > h(x),\\
\psi_0(x):&\quad (\exists x_0 < x)(\forall y \in (x, x_1))h(y) < h(x),\\
\psi_1(x):&\quad (\exists x_0 < x)(\forall y \in (x, x_1))h(y) = h(x),\\
\psi_2(x):&\quad (\exists x_0 < x)(\forall y \in (x, x_1))h(y) > h(x),\\
\theta_{ij}(x):&\quad \psi_i(x)\ \&\ \varphi_j(x).
\end{aligned}$$

Замечание 1 [1] *Если* $x \in Int(\mathrm{dom}(h))$, *то выполняется* $\theta_{ij}(x)$ *для некоторых* $i, j \in \{0,1,2\}$.

Утверждение 1 [1] *Пусть* I *— некоторый бесконечный интервал. Тогда ни одна из формул* $\theta_{01}, \theta_{10}, \theta_{12}, \theta_{21}$ *не может выполнятся на всем* I.

Таким образом, существенные случаи, которые необходимо рассматривать — это выполнимость на бесконечных интервалах формул θ_{22}, θ_{02}.

Очевидно, что в случае дискретного упорядочения леммы 1, 2 верны. Поэтому далее мы будем рассматривать только плотно упорядоченные структуры.

Лемма 1 *1-местная частичная функция, имеющая локальный минимум в каждой точке бесконечного выпуклого множества, не является определимой в слабо o-минимальной структуре.*

Доказательство, следующее ниже охватывает как случай функции из M в M, так и случай функции в сорт.

Доказательство будет проводится от противного:

Пусть M — слабо o-минимальная модель сигнатуры $\{<, f, \ldots\}$, где f — 1-местная частичная функция, определенная на бесконечном выпуклом множестве и имеющая в каждой точке этого множества локальный минимум. Приведем список свойств модели $\langle M, <, f\rangle$, из которых будет следовать противоречие.

На протяжении всего доказательства все рассматриваемые нами элементы и отношения лежат в $\mathrm{dom}(f)$.

Замечание 2 Можно считать, что f инъективна.

Доказательство. Определим отношение эквивалентности

$$E(x, y) \overset{def}{\Longleftrightarrow} f(x) = f(y).$$

Утверждение. Каждый класс эквивалентности по E конечен.

Доказательство. От противного. Пусть A — бесконечный класс, тогда, ввиду o-минимальности M, A содержит бесконечный открытый интервал I. Для произвольного элемента $a \in I$, т.к. f имеет локальный минимум в каждой точке, найдется $b \in I$ такой, что $f(b) > f(a)$. Противоречие.

Отсюда множество $Z \overset{def}{=} \{x : (\forall y)E(x, y) \to x \le y\}$ бесконечно. Поэтому можно считать, что $\mathrm{dom}(f) = Z$. □

Обозначения

- $U_a \overset{def}{=} \{x > a : (\forall y \in (a, x])f(y) > f(a)\} \cup \{x < a : (\forall y \in [x, a))f(y) > f(a)\} \cup \{a\}$.
- $a \prec b \overset{def}{\Longleftrightarrow} U_a \supset U_b$.
- $a \circ b \overset{def}{\Longleftrightarrow} a = b \vee a \prec b \vee b \prec a$.

Замечание 3 (i) U_a — выпуклое множество.

(ii) $a \ne b \Longrightarrow U_a \ne U_b$.

Свойство 1 *Если* $U_a \cap U_b \ne \emptyset$, *то либо* $U_a \subset U_b$, *либо* $U_b \subset U_a$.

Доказательство. Пусть $U_a \cap U_b \ne \emptyset$ и $a < b$. Рассмотрим случай, когда $f(a) < f(b)$. Если $b \in U_a$, то $U_b \subset U_a$. Если это не так, то существует d такой, что $U_a < d < b$ и $f(d) < f(a)$. Но тогда $f(d) < f(b)$ и $d \notin U_b$. Противоречие с тем, что $U_b \setminus U_a$ выпукло и U_b содержит некоторый элемент из U_a. □

Свойство 2 $\prec$ *— строгий частичный порядок.*

Свойство 3 *Для любой цепи* $a_0 \prec a_1 \prec \ldots \prec a_n$ *существует* $a_{n+1} \succ a_n$.

Доказательство. В качестве a_{n+1} можно взять любой элемент из U_{a_n}, который $\ne a_n$. □

Свойство 4 *Пусть $b \prec a, c \prec a$, тогда $b \diamond c$.*

Доказательство. Т.к. $b \prec a$ и $c \prec a$, то $U_a \subset U_b$ и $U_a \subset U_c$, следовательно $U_c \cap U_b \neq \emptyset$, и по свойству 1 либо $U_c \subset U_b$, т.е. $c \succ b$, либо $U_b \subset U_c$, т.е. $b \succ c$. □

Свойство 5 *Для любого a, множество $C_a \stackrel{\text{def}}{=} \{x : x \prec a\}$ конечно.*

Доказательство. Предположим, что C_a — бесконечное множество, тогда, ввиду слабой о-минимальности модели M, C_a содержит бесконечный открытый интервал I. Пусть d — произвольный элемент из I. Возьмем $m_1, m_2 \in I \cap U_d$ такие, что $m_1 < d < m_2$. Т.к. $m_1 \prec a$, $m_2 \prec a$, то по свойству 4 $m_1 \diamond m_2$. Предположим, что $m_1 \prec m_2$. Тогда $U_{m_2} \subset U_{m_1}$, т.е. $m_2 \in U_{m_1}$ и т.к. U_{m_1} выпукло, то $d \in U_{m_1}$, следовательно, $f(d) > f(m_1)$, а это противоречит $m_1 \in U_d$. □

Свойство 6 *$\prec$ — дискретный порядок.*

Доказателство является прямым следствием свойства 5. □

Свойство 7 *Для любых a, c таких, что $c \prec a$ найдется b со свойством $c \prec b \,\&\, \neg(a \diamond b)$.*

Доказательство. Возьмем в качестве b произвольный элемент из U_c, такой, что c лежит между a и b. □

Обозначения

- K — множество всех минимальных в смысле $\prec$ элементов.
- $\tilde{a} \stackrel{\text{def}}{=} \{x : a \prec x \,\&\, \neg(\exists y) a \prec y \prec x\}$

Свойство 8 *Множества $\tilde{a}$, по всем $a \in \mathrm{dom}(f)$, образуют равномерно определимое разбиение $\mathrm{dom}(f) \setminus K$.*

Доказательство. По определениям $\tilde{a}, K$, если $d \in K$, то $d \notin \tilde{b}$, для любого b.

Для $a \neq b$, $\tilde{a}$ не пересекается с $\tilde{b}$ по свойству 4. □

Свойство 9 *K конечно.*

Доказательство. Пусть K бесконечно, I — бесконечный открытый интервал, содержащийся в K. Возьмем $b \in I$, $c \in U_b \cap I$. Тогда $c \succ b$. Противоречие. □

Свойство 10 *Для любого a, $\tilde{a} \subseteq U_a$.*

Свойство 11 *Для любого a, $\tilde{a}$ конечно.*

Доказательство. Аналогично доказательству свойства 9. □

По разбиению из свойства 8 можно задать определимое отношение эквивалентности ε. Свойства 9, 11 означают, что ε — отношение эквивалентности с бесконечным числом конечных классов.

Рассмотрим бесконечное определимое множество X, состоящее из минимальных в смысле $<$ элементов классов эквивалентности по ε (т.е. $X = \{x : (\forall y)\varepsilon(x, y) \to x \leq y\}$).

В условиях предположения о слабой о-минимальности модели M мы можем разбить X на конечное число выпуклых множеств и выбрать среди них крайне правое множество U (т.е. $U \stackrel{\text{def}}{=} \{x \in X : (\forall y \in X) x < y \to [(\forall z) x < z < y \to z \in X]\}$).

Возьмем произвольный элемент $a \in U$, а для него найдем $b \succ a$ такой, что $b \notin X$ и $b > a$. Такой элемент b существует ввиду свойств 7, 3. Далее, возьмем произвольный $c \in U_b, c > b$, по свойству 3 $\tilde{c} \neq \emptyset$, следовательно, содержит элемент $d \in X$, а по свойству 10 $\tilde{c} > b$. Тогда имеем

$$a < b < d \,\&\, a, d \in X \,\&\, b \notin X.$$

Противоречие с выбором $a \in U$ доказывает лемму 1. □

Очевидно, что совершенно также доказывается несуществование определимой функции с локальным максимумом в каждой точке.

Теперь покажем, что если формула θ_{02} выполняется на открытом интервале I, то h — кусочно локально возрастающая функция на I.

Лемма 2 *В слабо о-минимальной модели M не является определимой функция g, удовлетворяющая следующим условиям в каждой точке x некоторого бесконечного выпуклого множества $U \subseteq \mathrm{dom}(g)$:*

(i) $(\exists z_1, z_2)(\forall v_1, v_2) z_1 < v_1 < x < v_2 < z_2 \to g(v_1) < g(x) < g(v_2)$,

(ii) $(\forall z)(\exists v_1, v_2) x < v_1 < v_2 < z \,\&\, g(v_1) \geq g(v_2)$.

Доказательство. Пусть существует такая функция g. Пусть для $a \in U$,

$$\begin{aligned} V_a &\overset{\text{def}}{=} \{x < a : (\forall y \in [x,a)) g(y) < g(a)\} \cup \\ &\cup \{x > a : (\forall y \in (a,x])) g(y) > g(a)\} \cup \{a\}, \\ f(a) &\overset{\text{def}}{=} \inf\{V_a\}. \end{aligned}$$

Заметим, что V_a является бесконечным (ввиду (i)) и выпуклым, а f — частичная функция из M в некоторый определимый сорт.

Следующее утверждение противоречит лемме 1.

Утверждение 2 *$f(x)$ определена и имеет локальный минимум в каждой точке U.*

Доказательство. Рассмотрим следующие формулы

$$\mu_0(x,a): \quad x < a \,\&\, f(x) \leq f(a), \tag{1}$$
$$\nu_0(x,a): \quad x > a \,\&\, f(x) \leq f(a), \tag{2}$$
$$\mu_1(x,a): \quad x < a \,\&\, f(x) > f(a), \tag{3}$$
$$\nu_1(x,a): \quad x > a \,\&\, f(x) > f(a). \tag{4}$$

Ввиду слабой o-минимальности M, для любого элемента $a \in U$ существуют интервалы $I < a < J$, для которых a является граничной точкой и которые полностью содержатся в множестве реализаций каких-то соответствующих формул из (1)–(4).

Утверждение. I не содержится в $\mu_0(M,a)$.

Доказательство. Пусть интервал I, имеющий a граничной точкой, полностью лежит в $\mu_0(M,a)$. Но тогда, рассматривая произвольную точку b из $V_a \cap I$, для которой, как известно, выполняется пункт (ii) леммы 2, мы найдем элементы c, d такие, что

$$b < c < d < a \,\&\, g(d) \leq g(c).$$

(Здесь b играет роль x, a — z, c — v_1, d — v_2 из (ii).)

Поэтому $f(d) > c > b$, а $b \in V_a$, т.е. мы получили, что $\not\models \mu_0(d,a)$, а это противоречит выбору $d \in I$. □

Утверждение. J не содержится в $\nu_0(M,a)$.

Доказательство. Аналогично доказательству предыдущего утверждения, для $b \in V_a \cap J$ найдутся c, d такие, что

$$a < c < d < b \,\&\, f(d) > c > a.$$

Следовательно, $d \notin \nu_0(M,a)$, а это противоречит выбору J. □

Таким образом лемма 2, а значит и теорема доказаны полностью.

Следствие *В любой слабо o-минимальной модели определимая частичная функция $f : M \to M$ имеет только конечное число точек разрыва.*

Литература

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TWO THEOREMS ON O-MINIMAL THEORIES

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1 O-minimal expansions[1]

Definition 1.1 [VdD] Model M of signature Σ is ordered minimal (o-minimal) if it is $\emptyset$-definable totally ordered and the realization of each formula of the signature $\Sigma(M)$ in one free variable is a disjoint union of finitely many of open intervals, points.

Throughout the section we consider only dense o-minimal structures.

Definition 1.2 Let A be a subset of totally ordered set B. Then A is convex if for any $\alpha, \beta \in A$ the following holds:

$$\forall \gamma \in B \ [\alpha < \gamma < \beta \Rightarrow \gamma \in A].$$

Note 1.1 An intersection of family of convex subsets of arbitrary totally ordered set is convex.

Definition 1.3 *(i) Let $M = \langle M, \Sigma \rangle$, where Σ is a signature of M, $\Sigma^+ \supset \Sigma$. Then $M^+ = \langle M, \Sigma^+ \rangle$ is said to be expansion of M.*

(ii) Expansion M^+ of model M is said to be essential if there exists $P^n \in \Sigma^+ \backslash \Sigma$ such that for any $\Phi(\bar{x}, \bar{a})$, $\bar{a} \in M$, $l(\bar{x}) = n$ it holds that $P(M^+)^n \neq \Phi(M^n, \bar{a})$.

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Definition 1.4 [P] *Let us say that theory T of signature Σ admits elimination of imaginaries if for all $M \in Mod(T)$, for all formula $\phi(\bar{x}, \bar{y})$ of signature Σ, for all $\bar{a} \in M^{l(\bar{y})}$ there exists $\bar{b} = \langle b_1, \ldots, b_n \rangle \in M^n$ such that $\forall M' \succ M$, $\forall f \in \mathrm{Aut}(M')$:*

$$\forall \bar{c} \in M' [\phi(\bar{c}, \bar{a}) \Leftrightarrow \phi(f(\bar{c}), \bar{a})] \iff \bigwedge_{i=1}^{n} f(b_i) = b_i$$

Definition 1.5 [D]

(i) A totally ordered model M of signature Σ is weakly ordered minimal (w.o.m.) if the realization of each formula of the signature $\Sigma(M)$ in one free variable is a disjoint union of finitely many convex subsets.

(ii) A theory T is said to be weakly ordered minimal (w.o.m.) if each model of T is w.o.m.

Note 1.2 T is w.o.m. iff for any formula $\Phi(x, \bar{y})$ of signature Σ there is a natural number $n_\Phi < \omega$ such that for any model of T for any $\bar{a} \in M^{l(\bar{y})}$, $\Phi(M, \bar{a})$ is an union $\leq n_\Phi$ $\neg\Phi(M, \bar{a})$-separable convex subsets of M. Here, $\Phi(M, \bar{a}) := \{b \in M : M \models \Phi(b, \bar{a})\}$.

Note 1.3 A set of all realizations of any 1-type over set of a model of w.o.m. theory in any model of this theory is convex set, because each complete type is determined by family of formulas, the realizations of which are convex (convex formula).

Theorem 1.1 *Let T be an o-minimal theory admitting elimination of imaginaires, $M \in Mod(T)$, M^+ is an o-minimal expansion of M. Then M^+ is essential expansion of M iff there exists elementary extension D^+ of M^+ such that the class of all D^+-definable partial unary functions does not coincide with the class of all D-definable partial unary functions.*

Proof. Sufficiency is evident.
Necessity.

Lemma 1.1 *Let D^+ be an ω-saturated elementary extension M^+. If each unary D^+-definable function is definable in D, then each D^+-definable set is definable in D.*

Proof of lemma 1.1. By induction on n we shall show, that

$(1)_n$ Each definable set in $(D^+)^n$ is definable in D.

$(2)_n$ Each partial D^+-definable function $f:(D^+)^n \to D$ is definable in D.

For $n=1$, $(2)_1$ is hypothesis, $(1)_1$ follows from $(2)_1$ and o-minimality of D^+.

$(1)_{n+1}$ Let $R \subset (D^+)^{n+1}$ is definable. Let S be projection of R on $(D^+)^n$.

$$\begin{aligned} S &:= \{\bar{a} \in (D^+)^n : D^+ \models \exists y(R(\bar{a},y))\} \\ S_\infty &:= \{\bar{a} \in S : D^+ \models \exists^\infty y(R(\bar{=}a,y))\} \\ S_m &:= \{\bar{a} \in (D^+)^n : D^+ \models \exists^{!m} y(R(\bar{a},y))\} \end{aligned}$$

S_∞ is definable in D^+ because in (weakly) o-minimal theory DIF (Definable Infinitary of Formula [B6]) property holds, i.e. $\forall\phi(x,\bar{y})\ \exists n_\phi$ such that $\forall\bar{y}\exists^{\geq n_\phi} x\phi(x,\bar{y}) \Rightarrow \exists^\infty x\phi(x,\bar{y})$.

By induction assumption $(1)_n$ S, S_m, S_∞ are definable in D^+

$$R_{i,m} := \{(\bar{a},b) \in R : \bar{a} \in S_m, 1 \leq i \leq m, b \text{ is } i\text{-th element belonging} R(\bar{a},D^+)\}$$

$R_{i,m}$ is the graph of definable in D^+ function, by induction assumption $(2)_n$ it is definable in D.

We consider $R' = S_\infty \times M - R$. R' is definable in D^+.

Let

$$\begin{aligned} P_m &:= \{\bar{a} \in (D^+)^n : D^+ \models \exists^{!m} y(R'(\bar{=}a,y))\} \\ P_\infty &= \{\bar{a} \in (D^+)^n : D^+ \models \exists^\infty y(R'(\bar{a},y))\} \\ P_{i,m} &= \{(\bar{a},b) : \bar{a} \in P_m, 1 \leq i \leq m, b \text{ is } i\text{-th element belonging} R'(\bar{a},D^+)\} \end{aligned}$$

$P_{i,m}$ is the graph of partial D^+-definable function, by induction assumption $(2)_n$ it is definable in D.

Then it is evident, that $S_\infty = P_1 \dot\cup P_2 \dot\cup \ldots \dot\cup P_N \dot\cup P_\infty$.

Consider P_∞. We define formula $\phi(\bar{x},y)$ as follows: for any $\bar{a}, b \in D^+$

$$\models \phi(\bar{a},b) \Leftrightarrow \models P_\infty(\bar{a}),$$

and b is a boundary point of set $R(\bar{a},D^+)$.

Fact, that the number of boundary points for any $\bar{a}$ is restricted by number k, follows from the DIF property for o-minimal theories [PS], [KPS].

Consider functions ϕ_i, ψ_i, $0 \leq i \leq k-1$

$$\begin{aligned} \phi_i(\bar{a}) = b &\Leftrightarrow b \in P_\infty \text{and } b \text{ is the beginning of } i\text{-th interval or } i\text{-th point in } R'(\bar{a},D^+) \\ \psi_i(\bar{a}) = b &\Leftrightarrow b \in P_\infty \text{and } b \text{ is the end } i\text{-th interval in } R'(\bar{a},D^+) \end{aligned}$$

It is evident, that ϕ_i, ψ_i are definable in D^+, and hence in D.

We denote

$$\begin{aligned} \theta_i(\bar{x},y) = {} & P_\infty(\bar{x}) \wedge \phi_i(\bar{x}) < y < \psi_i(\bar{x}) \wedge \\ & \wedge[R(\bar{x},\phi_i(\bar{x})) \to \phi_i(\bar{x}) \leq y < \psi_i(\bar{x})] \wedge \\ & \wedge[R(\bar{x},\psi_i(\bar{x})) \to \psi_i(\bar{x}) < y \leq \psi_i(\bar{x})] \wedge \\ & \wedge[R(\bar{x},\phi_i(\bar{x})) \wedge R(\bar{x},\psi_i(\bar{x})) \to= \phi_i(\bar{x}) \leq y \leq \psi_i(\bar{x})] \end{aligned}$$

Then

$$R(\bar{x},y) = \bigcup_{1\leq m\leq l, 1\leq i_m\leq m} R_{i_m,m}(\bar{x},y) \cup \bigcup_{i<k} \theta_i(\bar{x},y) \cup \bigcup_{1\leq m\leq N, 1\leq i_m\leq m+1} Q_{i,m}$$

where

$$Q_{i,m}(\bar{x},y) = P_m(\bar{x}) \wedge P_{i-1,m}(\bar{x}) < y < P_{i,m}(\bar{x}).$$

In our case $P_{0,m}(\bar{x}) = -\infty$, $P_{m+1,m}(\bar{x}) = \infty$.

So, set R is represented as a finite union of disjoint D-definable sets, hence R is D-definable set.

$(2)_{n+1}$ $f:(D^+)^{n+1} \to D^+$ is a partial definable function in D^+. Let $H(x) = \exists\bar{x}\exists y(f(\bar{x},x) = y)$. H is D-definable. For any $a \in H(D^+)$ the next function

$$f_a : (D^+)^n \to D^+,\ f_a(\bar{x}) = f(\bar{x},a)$$

is D-definable by induction assumption with the help of function $G_a(\bar{c},\bar{x})$, $\bar{c} \in D^+$, $G(\bar{y},\bar{x})$ is without parameters.

Claim 1.1 *There exists a finite number of functions* $G_1(\bar{x},\bar{y}_1),\dots,G_m(\bar{x},\bar{y}_m)$ *such that* $\forall a \in H(D^+)$ *there exists* $i \in \{1,\dots,m\}$, $\exists \bar{c}_a \in D^{l(\bar{y}_i)}$, *that*
$\forall \bar{x}[[G_i(\bar{x},\bar{c}_i) = f(\bar{x},a)] \wedge [G_i(\bar{x},\bar{c}_i)$ *is not definable* $\Leftrightarrow$
$\Leftrightarrow f(\bar{x},a)$ *is not definable*]].

Proof of claim 1.1 We assume contrary, i.e. there exists an infinite number of functions $\{G_i(\bar{x},\bar{y}_i) : i \in J\}$, that $\forall a \in H(D^+)$ there exists $G_j(\bar{x},\bar{c}_a)$, that $G_i(\bar{x},\bar{c}_a) = f(\bar{x},a)=$ and $\forall j \in J$ there exists suitable a_j such that $\forall j_1 \neq j,\ j_1 \in J$ the following is true

$$\models \forall \bar{y}[G_{j_1}(\bar{x},\bar{y}) \neq f(\bar{x},a_j)].$$

Then the following set is locally consistent with reference to $\mathrm{Th}(D^+,\bar{d})$, where $\bar{d} \in D^+$ is a tuple of elements taking part in definition $f(\bar{x},x)$

$$\{\forall \bar{y}_j(G_j(\bar{x},\bar{y}_j) \neq f(\bar{x},x) : j \in J)\} \cup \{H(x)\}.$$

Element $a \in D$, that satisfies this set of formulas, exists because D^+ is ω-saturated, that contradicts induction assumption.

Without loss of generality one may consider, that there exists the only function $G(\bar{y},\bar{x})$ (without parameters) of language of D such that $\forall a \in H(D^+)$ there exists $c_a \in D^+$, that $G(\bar{x},\bar{c}_a) = f(\bar{x},a)$.

We define $\bar{c} \sim \bar{c}',\ \bar{c},\bar{c}' \in D^k \Leftrightarrow \forall \bar{x}(G(\bar{x},\bar{c}) = G(\bar{x},\bar{c}'))$.

Existence of D-definable function $h : D^k \to D^j$ such that $\forall \bar{c},\bar{c}' \in D$

$$[h(\bar{c}) = h(\bar{c}') \Leftrightarrow G'(\bar{x},\bar{c}) = G(\bar{x},\bar{c}')$$

follows from the condition of theorem 1.1 on elimination of imaginaries.

Let $h_1(\bar{y}),\dots,h_j(\bar{y})$ be functions such that $\forall \bar{c} \in D^k,\ h(\bar{c}) =< h_1(\bar{y}),\dots,h_j(\bar{y}) >$. Define graphs of unary D^+-definable functions as follows:

$$K_i(x,z) = H(x) \wedge \exists \bar{y}(\forall \bar{x}(G(\bar{x},\bar{y}) = f(\bar{x},x) \wedge h_i(\bar{y}) = z)),\ i \in \{1,\dots,j\}.$$

$K_i(x,z)$ are D-definable by assumption of lemma. Let $d_i(x) = z \Rightarrow K_i(x,z)$. Then the following formula defines graph of D-definable $(n+1)$-ary function:

$$H(x) \wedge \exists \bar{y}(h(\bar{y}) =< d_1(x),\dots,d_j(x) > \wedge G(\bar{x},\bar{y}) = z).$$

One may check easily that it is graph of function. Lemma 1.1 is proved. □

Note. Analogous lemma for strongly minimal theories was proved by E. Hrushovsky [H], but only his condition is that D is expansion of algebraically closed field instead of condition on elimination of imaginaries. Our proof is based on the scheme of proof of Hrushovsky, only for another class of theories. One may note, that his proof also holds for strongly minimal theories admitting elimination of imaginaries. Taking in consideration results of B. Poizat [P], A. Pillay [AP1], V. Verbovsky [V], Tsuboi [T], one may formulate the following:

Hypothesis. Any strongly minimal theory admittes elimination of imaginaries iff it is expansion of theory of algebraically closed fields.

2 Number of formula copies in pairs of models

In [MMS] paper the Problem A (Cherlin) was formulated: will expansion of o-minimal structure by convex predicate be weakly o-minimal. For submodel $(M,+,\cdot,<,\dots,) \prec (R,+,\cdot,<,\dots,)$ they proved, that this expansion has weakly o-minimal theory. Dense submodel of R (i.e. M is contained in R densely) was considered. The main moment of induction step was the following:

Problem B For any formula $\phi(x,\bar{y},\bar{\alpha})$, $\alpha \in R \setminus M$ to find formula $K_\phi(\bar{y},\bar{\alpha})$ such that $\forall \bar{a} \in M^{l(\bar{y})}$ the following holds:

$$(M,R) \models \exists x \in M\ \phi(x,\bar{a},\alpha) \Longleftrightarrow R \models K_\phi(\bar{\ },\alpha).$$

In [B1] attempt to solve Problem A for o-minimal theories was undertaken. The following evident fact was applied for construction of the formula $K_\phi(\bar{y},\bar{\alpha},\bar{c})$.

Fact 2.1 *Let* $M \prec M'$, $\bar{\alpha},\bar{\beta} \in M' \setminus M$, $tp(\bar{\alpha}/M) = tp(\bar{\beta}/M)$. *For any formula* $\phi(x,\bar{a},\bar{\alpha})$, $\bar{a} \in M$ *the following holds:*

$$M' \models \exists x \in M\ \phi(x,\bar{a},\alpha) \Longrightarrow M' \models \exists x[\phi(x,\bar{a},\bar{\alpha}) \wedge \phi(x,\bar{a},\bar{\beta})].$$

The main difficulties in [B1] were the proofs, that:

(i) It is sufficient to consider finite number of copies of α over M.

(ii) Addition of new copies does not generate any irrationality.

In [BP] these difficulties were got over by application of Marker-Steinhorn's theorem on definability of types([MS]) in Pillay's formulation ([AP2]).

In [BP] an elegant proof based on the scheme of proof from [B1] was offered, however the number of copies of formula $\phi(x, \bar{a}, \bar{\alpha})$ was valued $2^{l(\bar{\alpha})}$.

B. Poizat asked the author about possible decrease of the number of copies of formula $\phi(x, \bar{a}, \bar{\alpha})$.

Convention. M' is sufficiently saturated of o-minimal theory.

Definition 2.1 Let $p \in S_1(A), B \subset M'$ such that M' is $|B \cup A|^+$-saturated. Then a neighbourhood of set B in the type p is the following set:

$$V_p(B) := \{\gamma \in M' | \exists \gamma_1, \gamma_2 \in p(M'), \exists H(x, \bar{b}, \bar{c}), \bar{b} \in B, \bar{c} \in A,$$
$$\gamma_1 < H(M', \bar{=}b, \bar{c}) < \gamma_2,\ \gamma \in H(M', \bar{b}, \bar{c})\}$$

. Let $\bar{\alpha} =< \alpha_1, \alpha_2, \ldots, \alpha_k >$ then $V_p(\bar{\alpha}) := V_p(\{\alpha_1, \alpha_2, \ldots, \alpha_k\})$.

Definition 2.2 (i) Let $p, q \in S_1(A)$. We say that p is orthogonal to q ($p \perp q$), if $\exists \alpha \in p(M')$ ($\equiv \forall \alpha \in p(M')$), $V_q(\alpha) = \emptyset$. If p is not orthogonal to q, then we denote this fact by $p \not\perp q$.

(ii) Let $B \subset M'$, $p \in S_1(a)$. We say B orthogonal to p ($B \perp p$) if $V_p(B) = \emptyset$.

Note 2.1 *(i) The relation $\not\perp$ is equivalence relation on 1-types.*

(ii) If $p \not\perp q$ then there is an A-definable bijection $f : p(M') \to q(M')$ such that f preserves $<$ or turn it over.

Definition 2.3 *Let $p \in S_1(A)$. We say that p is irrational type, if $p(M')$ has no limit points in $dcl(A)$.*

Note 2.2 [LM], [B1],[M] *Let $p, q \in S_1(A), p \not\perp q$. Then p is irrational iff q is irrational.*

Definition 2.4 *Let $p, q \in S_1(A), B \subset M'$ such that $p \perp q$, $B \perp p, B \perp q, \alpha \in p(M'), \beta \in q(M'), p' := tp(\alpha/A \cup B)$, $q' := tp(\beta/A \cup B)$. Then $p(M') = p'(M')$, $q(M') = q'(M')$. We say that p is B-orthogonal to q ($p(\perp, B)q$), if $p' \perp q'$. In opposite case, we say p is not B-orthogonal to q. We denote this fact by $p(\not\perp, B)q$.*

Note 2.3 *Let $B \subset M', A \subset M', O(B, S_1(A)) := \{p \in S_1(A) : B \perp p\}$. Then $(\not\perp, B)$ is equivalence relation on $O(B, S_1(A))$. This equivalence relation is stable under irrationality.*

Note 2.4 *Let $p, q \in S_1(A)$, $B \subset C_1$, $B \subset C_2$, $B \perp^{\omega} p$, $B \perp^{\omega} q$, $p(\not\perp^{\omega}, B)q$. Then the following is true:*

$$\exists \gamma (V_p(C_1) < \gamma < V_p(C_2) \quad or \quad V_p(C_2) < \gamma < V_p(C_1))$$

iff

$$\exists \eta (V_q(C_1) < \eta < V_q(C_2) \ \ or \ \ V_q(C_2) < \eta < V_q(C_1)).$$

Proof follows from Note 2.1 (ii).

Theorem 2.1 *Let M be an o-minimal model, M' be an elementary extension of M such that M' is $|M|^+$-saturated. Then for any $\bar{\alpha} \in M' \setminus M$ such that $tp(\alpha_m/M \cup \{\alpha_1, \ldots, \alpha_{m-1}\})$ is irrational there is $\bar{\beta} =< \bar{\beta}_1, \ldots, \bar{\beta}_{l(\bar{a})} >$, $tp(\bar{\beta}_m/M) = tp(\bar{\alpha}/M)$, $1 \le m \le l(\bar{\alpha})$ and for any $\phi(x, \bar{y}, \bar{\alpha})$, for any $\bar{a} \in M^{l(\bar{y})}$ the following holds:*

$$[\exists b \in M : M' \models \phi(b, \bar{a}, \bar{\alpha}) \iff M' \models \exists x (\bigwedge_{m=1}^{l(\bar{a})} \phi(x, \bar{a}, \bar{\beta}_m)).$$

Proof. It contains two stages:

(i) Choice of $\bar{\beta}$.

(ii) Conclusion.

(1) Choice of $\bar{\beta}$.

Let $\bar{\alpha} =< \alpha_1, \ldots, \alpha_n >$. Let $p_m = tp(\alpha_m/M \cup \bar{\alpha}|m-1), m \in \{1, 2, \ldots, n\}$.

Consider the sequence of one-types over $S_1(M)$, which is determinated by the following considerations:

$r_1 := p_1$.

Let $O_{m+1} := \{ r \in S_1(M) | \bar{\alpha}|m+1 \not\perp^{\omega} r, \bar{\alpha}|m \perp^{\omega} r \}$.

If $O_{m+1} = \emptyset$, then $r_{m+1} := r_m$, $\bar{\beta}_{m+1} := \bar{\beta}_m$.

If $O_{m+1} \neq \emptyset$, then r_{m+1} is arbitrary type from O_{m+1}.

Notice that, $\forall r \in O_{m+1}, r(\not\perp^{\omega}, \bar{\alpha}|m) r_{m+1}$ and r_{m+1} is irrational.

Let $\mu^1_{m+1}, \mu^2_{m+1} \in r_{m+1}(M')$ such that
$\mu^1_{m+1} < V_{r_{m+1}}(\bar{\alpha}) < \mu^2_{m+1}$.

Consider $fm+1 \in Aut_{M \cup \alpha|m}(M')$ such that $f_{m+1}(\mu^1_{m+1}) = \mu^2_{m+1}$.

Let $\bar{\beta}_{m+1} :=< f_{m+1}(\alpha_1), f_{m+1}(\alpha_2), \ldots, f_{m+1}(\alpha_n) >$. Then $tp(\bar{\beta}_{m+1}|M) = tp(\bar{\alpha}|M)$,
$\bar{\beta}_{m+1}|m = \bar{\alpha}|m$.

Let $< r_1, r_{i_2}, \ldots, r_{i_s} >$ be a sequence of all different types from $\bar{r} :=< r_1, r_2, \ldots, r_n >$, then $\bar{\beta} :=< \bar{\beta}_1, \bar{\beta}_{i_2}, \ldots, \bar{\beta} i_s >$.

(ii) Conclusion.

$\forall \phi(x, \bar{y}, \bar{\alpha}), \forall \bar{a} \in M^{l(\bar{y})}$ the following holds:

$$[\exists b \in M, M' \models \phi(b, \bar{a}, \bar{\alpha}) \Leftrightarrow M' \models \exists x(\phi(x, \bar{a}, \bar{\alpha}) \wedge \bigwedge_{j \in \{1, i_2, \ldots, i_s\}} \phi(x, \bar{a}, \bar{\beta}_j))].$$

Proof of Conclusion.

($\Rightarrow$) It is clear because $tp(\bar{\beta}_j/M) = tp(\bar{\alpha}/M)$.

($\Leftarrow$) Let $M' \models \exists x(\phi(x, \bar{a}, \bar{\alpha}) \wedge \wedge \wedge_{j \in \{1, i_2, \ldots, i_s\}} \phi(x, \bar{a}, \bar{\beta}_j))$.

Suppose $\phi(M', \bar{a}, \bar{\alpha}) \cap M = \emptyset, \phi(M', \bar{a}, \bar{\alpha}) = \cup^n_{k=1} \phi_k(M', \bar{a}, \bar{\alpha})$, where $\{\phi_k | k \leq n_\phi\}$ is maximal $\neg\phi(M', \bar{a}, \bar{\alpha})$-separable convex family of subformulas of $\phi(x, \bar{a}, \bar{\alpha})$ (Note 1.2).

Let $\phi_k(M', \bar{a}, \bar{\alpha})$ be arbitrary subformula. Because $\phi_k(M', \bar{a}, \bar{\alpha})$ is convex, then there is $r \in S_1(M')$ such that $\phi_k(M', \bar{a}, \bar{\alpha}) \subset r(M')$.

Let $j \in \{1, i_2, \ldots, i_s\}$ such that $r \in O_j$.
By Fact 2.1 and because $r(\not\perp^w, \alpha|j-1)r_j$ and r_j is irrational, r is irrational.

By choice of $\bar{\beta}_j$ and by Note 2.4 $V_r(\bar{\alpha}) \cap V_r(\bar{\beta}_j) = \emptyset$. Then $\phi_k(M', \bar{a}, \bar{\alpha}) \cap V_r(\bar{\beta}_j) = \emptyset$. So, $\phi(M', \bar{a}, \bar{\alpha}) \cap \wedge_{j \in \{1, i_2, \ldots, i_s\}} \phi(M', \bar{a}, \bar{\beta}_j) = \emptyset$.
Contradiction. Thus, $\phi(M', \bar{a}, \bar{\alpha}) \cap M \neq \emptyset$. □

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EQUATIONALLY COMPLETE CLASSES OF SEMIRINGS[1]

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1. Introduction

By a semiring we mean an algebra with two associative binary operations (denoted as addition and multiplication) such that the addition is commutative and the multiplication is distributive over the addition. The aim of the present short note is to find all minimal varieties (alias equationally complete classes, minimal primitive classes, etc.) of semirings. To that purpose, we shall make use of the following well known and easy observation:

1.1 Proposition. *Let $\mathcal{V}$ be a non-trivial variety of semirings. Then either $\mathcal{V}$ contains a (non-trivial) congruence-simple commutative semiring or every semiring from $\mathcal{V}$ is idempotent (i. e., both the addition and the multiplication are so).*

2. Some two-element semirings

$$S_1: \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \quad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\ ; \qquad S_2: \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array} \quad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}\ ;$$

$$S_3: \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \quad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\ ; \qquad S_4: \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \quad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}\ ;$$

$$S_5: \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \quad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}\ ; \qquad S_6: \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \quad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}\ ;$$

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$$S_7:\ \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \quad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 0 & 1 \end{array}\ ; \qquad S_8:\ \begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \quad \begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 1 & 1 \end{array}\ .$$

For $1 \le i \le 8$, let $\mathcal{S}_i$ denote the variety of semirings satisfying the following (semiring) equations:

$\mathcal{S}_1$ ... $\mathbf{x}+\mathbf{y}=\mathbf{uv}$;
$\mathcal{S}_2$ ... $\mathbf{x}+\mathbf{y}=\mathbf{u}+\mathbf{v}$, $\mathbf{xy}=\mathbf{yx}$, $\mathbf{x}=\mathbf{xx}$;
$\mathcal{S}_3$ ...... $\mathbf{x}=\mathbf{x}+\mathbf{x}$, $\mathbf{xy}=\mathbf{uv}$, $\mathbf{xy}=\mathbf{xy}+\mathbf{z}$;
$\mathcal{S}_4$ ...... $\mathbf{x}=\mathbf{x}+\mathbf{x}$, $\mathbf{xy}=\mathbf{uv}$, $\mathbf{z}=\mathbf{xy}+\mathbf{z}$;
$\mathcal{S}_5$ ...... $\mathbf{x}=\mathbf{x}+\mathbf{x}$, $\mathbf{x}=\mathbf{xx}$, $\mathbf{x}=\mathbf{xy}+\mathbf{x}$;
$\mathcal{S}_6$ ...... $\mathbf{x}=\mathbf{x}+\mathbf{x}$, $\mathbf{xy}=\mathbf{x}+\mathbf{y}$;
$\mathcal{S}_7$ ...... $\mathbf{x}=\mathbf{x}+\mathbf{x}$, $\mathbf{y}=\mathbf{xy}$;
$\mathcal{S}_8$ ...... $\mathbf{x}=\mathbf{x}+\mathbf{x}$, $\mathbf{x}=\mathbf{xy}$.

2.1 Proposition. *(i) Every non-trivial semiring from $\mathcal{S}_1$ is a subdirect product of copies of S_1.*

(ii) $\mathcal{S}_1$ is a minimal variety.

Proof. (i) If $S \in \mathcal{S}_1$ is subdirectly irreducible, then every congruence of $S(+)$ is also a congruence of $S(\cdot)$ and consequently $S(+)$ is a subdirectly irreducible semilattice, $S(+) \cong S_1(+)$ and $S \cong S_1$.

(ii) This is an immediate consequence of (i).

2.2 Proposition. *(i) Every non-trivial semiring from $\mathcal{S}_2$ is a subdirect product of copies of S_2.*

(ii) $\mathcal{S}_2$ is a minimal variety.

Proof. (i) If $S \in \mathcal{S}_2$ is subdirectly irreducible, then every congruence of $S(\cdot)$ is also a congruence of $S(+)$ and we have $S(\cdot) \cong S_2(\cdot)$. Now, if $o = x + y$, $x, y \in S$, then $xo = x(x+x) = x + x = o$. Consequently, o is the absorbing element of $S(\cdot)$ and $S \cong S_2$.

2.3 Proposition. *(i) Every non-trivial semiring from $\mathcal{S}_3$ is a subdirect product of copies of S_3.*

(ii) $\mathcal{S}_3$ is a minimal variety.

Proof. (i) If $S \in \mathcal{S}_3$ is subdirectly irreducible, then every congruence of $S(+)$ is a congruence of $S(\cdot)$, and hence $S(+) \cong S_3(+)$. Now, if $o = xy$, $x, y \in S$, then $o = o + z$ for every $z \in S$, and so o is the absorbing element of $S(+)$ and $S \cong S_3$.

2.4 Proposition. *(i) Every non-trivial semiring from $\mathcal{S}_4$ is a subdirect product of copies of S_4.*

(ii) $\mathcal{S}_4$ is a minimal variety.

Proof. Similar to that of 2.3 (use the fact that the absorbing element of $S(\cdot)$ is a neutral element of $S(+)$).

2.5 Proposition. *(i) Every non-trivial semiring from $\mathcal{S}_5$ is a subdirect product of copies of S_5.*

(ii) $\mathcal{S}_5$ is a minimal variety.

Proof. In fact, $\mathcal{S}_5$ is the variety of distributive lattices.

2.6 Proposition. *(i) Every non-trivial semiring from $\mathcal{S}_6$ is a subdirect product of copies of S_6.*

(ii) $\mathcal{S}_6$ is a minimal variety.

Proof. $\mathcal{S}_6$ is equivalent to the variety of semilattices.

2.7 Proposition. *(i) Every non-trivial semiring from $\mathcal{S}_7$ (or $\mathcal{S}_8$) is a subdirect product of copies of S_7 (or S_8).*

(ii) Both $\mathcal{S}_7$ and $\mathcal{S}_8$ are minimal varieties.

Proof. Easy.

3. Zero-multiplication rings

For a prime $p \in \mathbb{P}$ (the set of all prime numbers), let $\mathbb{Z}_{p,0}$ denote the zero-multiplication ring on the group $\mathbb{Z}_p(+)$ of integers mod p. Further, let $\mathcal{Z}_p$ be the variety of semirings satisfying the equations $p\mathbf{x} = p\mathbf{y}$, $\mathbf{x} = (p+1)\mathbf{x}$, $p\mathbf{x} = \mathbf{yz}$.

3.1 Proposition. *(i) Every non-trivial semiring from $\mathcal{Z}_p$ is a subdirect product of copies of $\mathbb{Z}_{p,0}$.*

(ii) $\mathcal{Z}_p$ is a minimal variety.

Proof. Obvious.

4. Fields

For $p \in \mathbb{P}$, let $\mathbb{Z}_{p,1}$ denote the field of integers mod p. Further, let $\mathcal{F}_p$ be the variety of semirings satisfying the equations $px = py$, $x = (p+1)x$, $x = x^p$, $xy = yx$.

4.1 Proposition. *(i) Every non-trivial semiring from $\mathcal{F}_p$ is a subdirect product of copies of $\mathbb{Z}_{p,1}$.*

(ii) $\mathcal{F}_p$ is a minimal variety.

Proof. (i) Let $S \in \mathcal{F}_p$, $S \neq 0$. Since $a^p = a$ for every $a \in S$, S is a semiprime ring and S is a subdirect product of prime rings from $\mathcal{F}_p$. Now, if S is prime, then $a^p = a$ implies that S is a field and $S \cong \mathbb{Z}_{p,1}$.

5. Congruence–simple commutative semirings

Let G $(= G(\cdot))$ be a (multiplicatively denoted) abelian group. We shall define a semiring $V(G)$ in the following way:

(a) The underlying set of $V(G)$ is $G \cup \{o\}$, where $o \notin G$;

(b) G is multiplicative subgroup of $V(G)$;

(c) o is an absorbing element of $V(G)$;

(d) $a + b = o$ for all $a, b \in V(G)$, $a \neq b$;

(e) $a + a = a$ for every $a \in V(G)$.

Let A be a subsemigroup of the additive group $\mathbb{R}(+)$ of real numbers. We shall define a semiring $W(A)$ in the following way:

(a) The underlying set of $W(A)$ is A;

(b) The addition $\oplus$ of $W(A)$ is given by $a \oplus b = \min(a, b)$;

(c) The multiplication $*$ of $W(A)$ is given by $a * b = a + b$ (the addition of reals).

The following classification of congruence-simple commutative semirings can be found in [1]:

5.1 Proposition. *Let S be a congruence-simple commutative semiring (i. e., S possesses exactly two congruences). Then at least one of the following six cases takes place:*

(1) $S \cong S_1$, S_2, S_3, S_4, S_5.

(2) $S \cong V(G)$ for an abelian group G.

(3) $S \cong W(A)$ for a subsemigroup A of $\mathbb{R}(+)$.

(4) S is isomorphic to a subsemiring of the semiring $\mathbb{R}^+$ of positive real numbers.

(5) S is a zero-multiplication ring of finite prime order.

(6) S is a field.

6. Minimal varieties of semirings

6.1 Theorem. *The (pair-wise) different varieties S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, $\mathcal{Z}_p$, $\mathcal{F}_p$, $p \in \mathbb{P}$, are just all minimal varieties of semirings.*

Proof. All the mentioned varieties are minimal according to 2.1, …, 2.7, 3.1, 4.1. Now, conversely, let $\mathcal{V}$ be a minimal variety of semirings. In view of 1.1 and 5.1, we have to distinguish the following cases:

(i) $S_i \in \mathcal{V}$ for some $1 \leq i \leq 5$.

Then $\mathcal{V} = \mathcal{S}_i$.

(ii) $V(G) \in \mathcal{V}$ for an abelian group G.

Then $\mathcal{V} = \mathcal{S}_6$, since S_6 is isomorphic to a subsemiring of $V(G)$.

(iii) $W(A) \in \mathcal{V}$ for a non-trivial subsemigroup A of $\mathbb{R}(+)$.

Now, take $a \in A$ such that $a > 0$ (or $a < 0$) and put $B = \{na; n = 1, 2, 3, \dots\}$, $C = \{na; n = 2, 3, 4, \dots\}$ and $r = (C \times C) \cup \mathrm{id}_B$. Then B is a subsemiring of $W(A)$, $B \in \mathcal{V}$, r is a congruence of B and $B/r \cong S_4$ (or $B/r \cong S_3$). Thus $\mathcal{V} = \mathcal{S}_4$ (or $\mathcal{V} = \mathcal{S}_3$), which is a contradiction.

(iv) $R \in \mathcal{V}$, where R is a subsemiring of $\mathbb{R}^+$.

Now, take $a \in R$, $a > 1$, and denote by A the subsemiring of $\mathbb{R}^+$ generated by a. Then $A \in \mathcal{V}$ and $r = (B \times B) \cup \mathrm{id}_A$ is a congruence of A, where $B = \{b \in A; b > a\}$. Since $A/r \cong S_1$, we have $\mathcal{V} = \mathcal{S}_1$, a contradiction.

(v) $\mathbb{Z}_{p,0} \in \mathcal{V}$, $p \in \mathbb{P}$.

Then $\mathcal{V} = \mathcal{Z}_p$.

(vi) $F \in \mathcal{V}$, F being a field.

Then there is $p \in \mathbb{P}$ such that $\mathbb{Z}_{p,1} \in \mathcal{V}$ and we have $\mathcal{V} = \mathcal{F}_p$.

(vii) Every semiring from $\mathcal{V}$ is idempotent.

Now, let $S \in \mathcal{V}$ be generated by two elements, say a and b. Then every element from S is a sum of the following elements: a, b, ab, ba, aba, bab.

If $c = ab \neq aba = d$, then the subsemiring generated by c, d belongs to $\mathcal{S}_7$ and consequently $\mathcal{V} = \mathcal{S}_7$. Similarly, if $ba \neq bab$, then $\mathcal{V} = \mathcal{S}_8$.

If $c = ab = aba \neq ba = bab = e$, then $ce = c$, $ec = c$ and $\mathcal{V} = \mathcal{S}_8$.

Finally, if $ab = ba$, then S is commutative and $\mathcal{V} = \mathcal{S}_5, \mathcal{S}_6$.

6.2 Corollary. *Every minimal variety of semirings is finitely based.*

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О ПОДАЛГЕБРАХ ИТЕРАЦИОННЫХ АЛГЕБР РЕКУРСИВНЫХ ФУНКЦИЙ

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Интерес к счетным классам теоретико-числовых функций, замкнутых относительно операций суперпозиции $*$ и разновидностей операции итерации связан, прежде всего, с изучением основных классов вычислимых функций. Алгебры примитивно-рекурсивных и обще-рекурсивных 1-местных функций указанного типа изучались в [1,2]. Известно, что операции примитивной и общей рекурсии (частичная операция) связаны с основными операторами языков программирования типа "for - do" и "while - do" и до сих пор являются предметом пристального изучения, так в [6] рассматривались следующие операции над вычислимыми функциями:

а) примитивная рекурсия $(f|g)(x) = g^{f(x)}(x)$;
б) шаговая рекурсия $(f \downarrow g)(x) = \mu m[f(g^{(m)}(x) = 0]$;
в) общая рекурсия $(f/g)(x) = g^{(f \downarrow g)(x)}(x)$.

Зафиксируем некоторый счетный, замкнутый относительно $*$ класс $\mathcal{N}$ 1-местных всюду определенных теоретико-числовых функций, и все замкнутые классы, определяемые далее, будем рассматривать в качестве подклассов класса $\mathcal{N}$. Первый, традиционный способ описания замкнутых классов — задание их в виде идеалов полугруппы преобразований.

Утверждение 1. Следующие замкнутые классы являются идеалами полугруппы $\mathcal{N}$:

$$\left.\begin{array}{l} \text{а) } \mathcal{I}_m = \{f \mid |\rho f| \leq M\},\ m = 1, 2, \ldots \\ \text{б) } \mathcal{I}_\omega = \cup \mathcal{I}_m \end{array}\right\} \text{— двусторонние идеалы,}$$

$$\left.\begin{array}{l}\text{в) } \mathcal{R}_1 = \{f|\overline{\rho f} \text{ бесконечно}\} \\ \text{г) } \mathcal{R}_2 = \{f|\overline{\rho f} \neq \emptyset\}\end{array}\right\} \text{— правые идеалы,}$$

$$\left.\begin{array}{l}\text{д) } \mathcal{L}_1 = \{f|(\exists a, b)\, a \neq b \,\&\, f(a) = f(b)\} \\ \text{е) } \mathcal{L}_2 = \{f|(\forall a \in \rho f)\, f^{-1}(a) \text{ бесконечно}\}\end{array}\right\} \text{— левые идеалы.}$$

Доказательство: очевидно следует из определения классов.

Заметим, что элементами $\mathcal{I}_1$ являются все функции-константы, элементами $\mathcal{I}_\omega$ — все ограниченные функции, элементами $\mathcal{L}_1$ — все функции, не являющиеся 1-1-функциями, элементами $\mathcal{L}_2$ — все функции большого размаха и т.д.

Пусть S — некоторая счетная решетка подмножеств натурального ряда. Замкнутые подклассы класса $\mathcal{N}$ можно определять тогда путем отнесения к некоторому идеалу (фильтру) решетки S тех или иных "характеристических" множеств. В качестве таких множеств берутся ρf, $\eta f = \{x|f(x) = x\}$, $f^{-1}(a)$, а также множества $\alpha f = \{a|f^{-1}(a) \text{ бесконечно}\}$, $\beta_m f = \{x|m \cdot f(x) > x\}$, $Sf = \{x|(\forall y < x)\, f(y) \neq f(x)\}$, $\sigma f = \bigcup\limits_{m \in N} \sigma_m f$, где $\sigma_m f = \emptyset$, если $f(m) \leq m$ и $\sigma_m f = \{m, m+1, \ldots, f(m) - 1\}$ если $f(m) > m$.

Утверждение 2. Пусть $\mathcal{J}(\mathcal{F})$ — некоторый идеал (фильтр) решетки S, тогда следующие множества функций являются замкнутыми подклассами класса $\mathcal{N}$:

а) $\mathcal{I}_\mathcal{J} = \{f|\rho f \in \mathcal{J}\}$;

б) $\bar{\mathcal{I}}_\mathcal{J} = \{f|\overline{\rho f} \text{ конечно}\}$;

в) $\mathcal{N}^1_\mathcal{J} = \{f|(\forall m > 0)\, \beta_m f \text{ конечно}\}$;

г) $\mathcal{N}^2_\mathcal{J} = \{f|Sf \in \mathcal{J}\}$;

д) $\mathcal{N}^3_\mathcal{J} = \{f|\alpha f \text{ конечно}\}$;

е) $\mathcal{N}^4_\mathcal{J} = \{f|(\forall a) f^{-1}(a) \text{ конечно}\}$;

ж) $\mathcal{N}^5_\mathcal{J} = \{f|\sigma f \in \mathcal{J}\}$;

з) $\mathcal{N}^1_\mathcal{F} = \{f|\nu f \in \mathcal{F}\}$.

Доказательство. а) $\rho(f * g) \subseteq \rho f \Rightarrow \rho(f * g) \in \mathcal{J}$;

б) $\rho(f * g)$ конечно в силу конечности $\overline{\rho f}$ и $\overline{\rho g}$;

в) конечность $\beta_m(f * g)$ следует из конечности $\beta_m f$;

г) $S(f * g) \subseteq Sg \Rightarrow S(f * g) \in \mathcal{J}$;

д) следует из того, что $\alpha(f * g) \subseteq \alpha f \cup f(\alpha g)$;

е) множество $(f * g)^{-1}(a)$ конечно как конечное объединение конечных множеств;

ж) если $\sigma(f * g) = \emptyset$, то $f * g \in \mathcal{N}^5_\mathcal{J}$. Пусть $x \in \sigma(f * g)$, т.е. $(\exists k \leq x)\, f(g(k)) > k$ и пусть $x \in \sigma_k(f * g)$. Если $x < g(k)$, то $x \in \sigma_k g$, а при $x \geq g(k)$ $x \in \sigma_{g(k)} f$, иначе говоря, всегда $\sigma(f * g) \subseteq \sigma f \cup \sigma g$;

з) $\nu(f * g) \supseteq \nu f \cap \nu g \Rightarrow \nu(f * g) \in \mathcal{F}$.

Отметим, что в топологической терминологии класс $\mathcal{N}^1_\mathcal{F}$ есть росток функции $I(x) = x$ относительно фильтра $\mathcal{F}$. Очевидно, что замыкание $\{I\}$ есть $\{I\}$, т.е. $\{I\}$ — минимальный подкласс $\mathcal{N}$. Для всякого $m \in N$ замыкание функции-константы $c_m = m$ тоже есть минимальный подкласс и для него имеет место аналогичный факт (см. [1]).

Еще один традиционный способ задания замкнутых классов связан с понятием инварианта. Пусть R — произвольное конечно- или счетноместное отношение на N. Говорят, что функция $f \in \mathcal{N}$ сохраняет отношение R, если $f(R) \subseteq R$. В этом случае R называется инвариантным для f или инвариантом f. R называется инвариантом подкласса $\mathcal{N}_1 \subset \mathcal{N}$, если R инвариант f для всякой $f \in \mathcal{N}_1$.

Рассмотрим вопрос об описании отношений, инвариантных для функций из произвольного замкнутого подкласса $\mathcal{N}_0 \subset \mathcal{N}$, аналогично тому, как это сделано для функций k-значной логики [4]. Для этого определим некоторую алгебру отношений со следующими операциями.

1) *Переименование координат.* Всякое отношение R есть, вообще говоря, счетное множество точек, каждая из которых является, вообще говоря, счетной последовательностью элементов N. Каждую такую последовательность мы будем называть столбцом отношения R, а последовательность i-х координат, соответственно, i-й строкой отношения R и обозначим i-ю строку через R_i. Пусть p — произвольная перестановка N, тогда переименование координат, примененное к отношению R, дает новое отношение pR, которое определяется построчно: $(pR)_i = R_{p(i)}$.

2) *A-проекция или A-цилиндрификация.* Пусть A — произвольное подмножество N и $a(n)$ — прямой пересчет A. Тогда A-проекция R_A отношения R определяется построчно: $(R_A)_i = R_{a(i)}$.

3) *Дублирование.* Пусть A — произвольное подмножество N и R_j — произвольная строка отношения R, тогда все строки нового отношения $d^j_A R$ с координатами из A совпадают с R_j, т.е. $(d^j_A R)_a = R_j$, $a \in A$, а строки с координатами из $\overline{A}$ равны соответствующим строкам R:

$$(d^j_A R)_b = R_b,\ b \in \overline{A}.$$

4) *Объединение* и 5) *Пересечение.* Если $R_0, R_1, \ldots$ — последовательность отношений,то для любого $A \in N$ обычным образом определяются $\bigcup\limits_{a \in A} R_a$, $\bigcap\limits_{a \in A} R_a$.

Выделенными элементами в алгебре отношений являются диагонали: пусть E — некоторое отношение эквивалентности на N, сопоставим ему отношение DE на N такое, что $mEn \Rightarrow (DE)_{mj} = (DE)_{nj}$ для всех j. (Здесь и далее R_{ij} обозначает j-й элемент i-й строки отношения R).

Утверждение 3. Совокупность инвариантов всякого замкнутого подкласса $\mathcal{N}_1 \subset \mathcal{N}$ есть подалгебра алгебры отношений.

ДОКАЗАТЕЛЬСТВО. 1) Пусть R — произвольный инвариант $\mathcal{N}_1$, тогда если r - произвольная точка (столбец) R, f — некоторая функция из $\mathcal{N}_1$, то мы имеем: $f(p(r)) = (f * p)(r)$, откуда следует инвариантность отношения pR.
2) Если $r \in R_A$, то $f(r) \in R_A$ по определению A-проекции и в силу того, что R — инвариант для f.
3) Очевидно из определения операции дублирования.
4) Пусть $r \in \cup R_i$ и все R_i — инварианты для произвольной $f \in \mathcal{N}_1$, тогда, очевидно $f(r)$ принадлежит R_i: для некоторого i, следовательно, $\cup R_i$ также инвариант для f.
5) Если $r \in \cap R_i$, то $f(r) \in R_i$ для всех i, значит, $\cap R_i$ — инвариант для f.
6) Диагональ является инвариантом всякой функции f, поскольку из

$R_{mj} = R_{nj}$ для всякого j следует $f(R_{mj}) = f(R_{nj})$ для всякого j.

Покажем теперь,что всякий инвариант подкласса $\mathcal{N}_1$ получается из некоторого фиксированного инварианта с помощью введенных выше операций. Для этого определим понятие графика подкласса, аналогичное понятию 1-графика из [4] и обобщающее в некотором смысле понятие m-n-спиннтера из [5]. Пусть $f_0 = I, f_1, f_2, \ldots$ — функции, входящие в подкласс $\mathcal{N}_1$. Тогда график $\Gamma_{\mathcal{N}_1}$ подкласса $\mathcal{N}_1$ есть счетная матрица, причем j-й столбец матрицы $\Gamma_{\mathcal{N}_1}$ есть последовательность чисел $f_j(0), f_j(1), f_j(2), \ldots$. Очевидно, график всякого подкласса задается с точностью до порядка столбцов.

Утверждение 4. Всякий инвариант подкласса $\mathcal{N}_1$ получается из $\Gamma_{\mathcal{N}_1}$ операциями 1) – 5).

ДОКАЗАТЕЛЬСТВО. Пусть $R \subset N^m$ $(m \geq 1)$ — произвольный m-арный инвариант для $\mathcal{N}_1$, тогда $R = \bigcup\limits_{r \in R} (\Gamma_{\mathcal{N}_1})_{\{r\}}$. Если же $R \in N^N$ — счетноарный инвариант $\mathcal{N}_1$, то сначала операциями A-проекции, дублирования и переименования строк получим из $\Gamma_{\mathcal{N}_1}$ произвольный столбец $r_0 \in R$, а затем снова имеем $R = \bigcup\limits_{r \in R} (\tilde{\Gamma}_{\mathcal{N}_1})_{\{r\}}$, где $\tilde{\Gamma}_{\mathcal{N}_1}$ — результат указанного преобразования графика $\Gamma_{\mathcal{N}_1}$.

Заметим, что если интересоваться важным случаем рекурсивно-перечислимых бинарных отношений, то как показано в [7], при некотором естественном выборе операций алгебра указанных отношений является конечно-порожденной.

В связи с соответствиями Галуа естественно возникает вопрос об описании эндоморфизмов данной алгебры отношений, т.е. функций, для которых отношения из данной алгебры являются инвариантами (см. [4]). Для случая рекурсивных функций, которые здесь прежде всего рассматриваются, обратное соответствие Галуа не имеет места, поскольку нетрудно указать пример алгебры отношений, не все эндоморфизмы которой рекурсивны.

Отметим также следующий очевидный факт.

Утверждение 5. Пусть T — произвольное транзитивное отношение на N, тогда $\mathcal{N}_T = \{f | \Gamma_f \subseteq T\}$ есть замкнутый подкласс $\mathcal{N}$.

В частности, в [1] рассмотрены важные частные случаи отношений эквивалентности и частичного порядка для 1-местных примитивно-рекурсивных функций.

Возвращаясь к операциям примитивной рекурсии | и общей рекурсии /, получим из приведенных выше утверждений следующее

Следствие. Пусть $\mathcal{N}_p$ и $\mathcal{N}_R$ получены из $\mathcal{N}$ добавлением к сигнатуре операций | и / соответственно. Тогда подклассы $\mathcal{I}_m, \mathcal{I}'_\omega, \mathcal{R}_1, \mathcal{R}_2, \mathcal{L}_1, \mathcal{I}_{\mathcal{J}}, \tilde{\mathcal{I}}_{\mathcal{J}}, \mathcal{N}^1_{\mathcal{J}}, \mathcal{N}^2_{\mathcal{J}}, \mathcal{N}^1_{\mathcal{F}}$ и $\mathcal{N}_T$ являются также подалгебрами в $\mathcal{N}_p$ и N_R (с учетом частичности операции /).

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ON n-PERMUTABILITY OF VARIETES HAVING p-DETERMINED CONGRUENCES

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The concept of p-determined congruences was introduced by J. Słominski [6]:

Definition. Let $A = (A, F)$ be an algebra of type τ and $p(x,y)$ be a binary term of type τ. A congruence $\Theta \in Con A$ is called *p-determined* if there exists an element $d \in A$ such that

$$\langle a,b\rangle \in \Theta \quad \text{if and only if} \quad \langle p(a,b), p(d,d)\rangle \in \Theta.$$

A has *p-determined congruences* if each $\Theta \in Con A$ is p-determined. A variety $\mathcal{V}$ has *p-determined congruences* if each $A \in \mathcal{V}$ has this property.

Algebras having p-determined congruences were treated in [5], [6] and Mal'cev condition characterizing varieties having p-determined congruences can be found in [1].

Following [4], an algebra A is *n-permutable* if $\Theta \circ \Phi \circ \Theta \circ \ldots = \Phi \circ \Theta \circ \Phi \circ \ldots$ holds for every $\Theta, \Phi \in Con A$, where we have n factors in each side of this equality. A is *permutable* if it is 2-permutable. A variety $\mathcal{V}$ is *n-permutable* or *permutable* if every $A \in \mathcal{V}$ has this property.

Theorem 1 *If a variety $\mathcal{V}$ has p-determined congruences then $\mathcal{V}$ is n-permutable for some $n \geq 2$.*

Proof. Let $\mathcal{V}$ be a variety of type τ, let $p(x,y)$ be a binary term of type τ and suppose that $\mathcal{V}$ has p-determined congruences. Denote by A the free algebra $F_{\mathcal{V}}(x,y)$ of $\mathcal{V}$ generated by two free generators x, y. Since the least congruence $\omega_{\mathcal{A}} \in Con\mathcal{A}$ (i.e. the identity relation on $\mathcal{A}$) is p-determined, we have $(p(x,x), p(y,y)) \in \omega_{\mathcal{A}}$, i.e.

$$p(x,x) = p(y,y). \tag{1}$$

By the definition, we have $(x,y) \in \Theta(p(x,y), p(x,x))$, where $\Theta(a,b)$, denotes the least congruence of $Con\mathcal{A}$ containing the pair (a,b). In account of Mal'cev lemma, there exist binary polynomials $\varphi_1, \dots, \varphi_n$ over $\mathcal{A}$ such that $x = \varphi_1(p(x,y), p(x,x))$,

$$\varphi_i(p(x,x), p(x,y)) = \varphi_{i+1}(p(x,y), p(x,x)) \text{ for } i = 1, 2, \dots, n-1 \tag{2}$$

$$y = \varphi_n(p(x,x), p(x,y)).$$

However, $\mathcal{A}$ is the free algebra with two free generators x, y, i.e. there exist 4-ary terms $t_1, \dots, t_n$ with

$$\varphi_i(z,v) = t_i(z,v,x,y) \quad \text{for} \quad i = 1, 2, \dots, n. \tag{3}$$

We can set $q_0(x,y,z) = x$, $q_{n+1}(x,y,z) = z$ and

$$q_i(x,y,z) = t_i(p(x,y), p(y,z), x, z).$$

It is an easy exercise to verify that $(1), (2), (3)$ yield $x = q_0(x,y,z)$,

$$q_i(x,x,z) = q_{i+1}(x,z,z), \quad i = 0, \dots, n-1$$

$$z = q_n(x,y,z)$$

which is the well-known Mal'cev condition characterizing $(n+1)$-permutable varieties $(n \geq 1)$ derived by J. Hagemann and A. Mitschke. ¶

By a *tolerance* on an algebra $\mathcal{A} = (A, F)$ is meant a reflexive and symmetric binary relation on A having the substitution property with respect to F (i.e. it is a subalgebra of the direct square $\mathcal{A} \times \mathcal{A}$). Thus every congruence on $\mathcal{A}$ is a tolerance on $\mathcal{A}$ (but not vice versa in a general case). If also conversely every tolerance on $\mathcal{A}$ is a congruence on $\mathcal{A}$, $\mathcal{A}$ is called *tolerance trivial algebra*. A variety $\mathcal{V}$ is *tolerance trivial* if every $\mathcal{A}$ of $\mathcal{V}$ has this property. It was shown by the first author (see e.g. [3]) that a variety $\mathcal{V}$ is tolerance trivial

if and only if $\mathcal{V}$ is permutable. Hence, every variety of groups, rings, modules or the variety of Boolean algebras are examples of tolerance trivial varieties.

Denote by $Tol\mathcal{A}$ the set of all tolerances on an algebra $\mathcal{A}$. It was shown (see e.g. [3]) that $Tol\mathcal{A}$ in an algebraic lattice with respect to set inclusion; its least element is ω_A and the greatest one is $A \times A$. Moreover, for any a,b of A there exists the least tolerance of A containing the pair (a,b); it will be denoted by $T(a,b)$. As it was shown (see e.g. Lemma 1.7 in [3]), $(c,d) \in T(a,b)$ if and only if there exists a binary polynomial ψ over $\mathcal{A}$ such that $c = \psi(a,b)$ and $d = \psi(b,a)$.

The following concept was introduced in [2] :

Let $\mathcal{A} = (A, F)$ be an algebra of type τ and $p(x,y)$ be a binary term of type τ. A tolerance $T \in Tol\mathcal{A}$ is *p-determined* if there exists $d \in A$ such that

$$\langle a,b \rangle \in T \text{ if and only if } \langle p(a,b), p(d,d) \rangle \in T.$$

$\mathcal{A}$ *has p-determined tolerances* if every T of $Tol\mathcal{A}$ is p-determined. A variety $\mathcal{V}$ *has p-determined tolerances* if each $\mathcal{A} \in \mathcal{V}$ has this property.

Example. Consider a groupoid $\mathcal{A} = (\{a,b,c\}, \cdot)$ whose operation is given by the table

$\cdot$	a	b	c
a	a	b	c
b	b	b	b
c	c	b	a

Let $T = \{\langle a,a \rangle, \langle b,b \rangle, \langle c,c \rangle, \langle a,b \rangle, \langle b,a \rangle, \langle b,c \rangle, \langle c,b \rangle\}$. It is a routine way to verify that $T \in Tol\mathcal{A}$. Put $p(x,y) = x \cdot y$ and take $d = a$. Then T is p-determined since $\langle x,y \rangle \in T$ if and only if $\langle x \cdot y, a \cdot a \rangle \in T$.

Theorem 2 *A variety $\mathcal{V}$ has p-determined tolerances if and only if $\mathcal{V}$ is permutable and has p-determined congruences.*

Proof. If $\mathcal{V}$ is permutable and has p-determined congruences, then $\mathcal{V}$ is tolerance trivial (see [3]) thus every tolerance on $\mathcal{A} \in \mathcal{V}$ is a congruence on $\mathcal{A}$, i.e. $\mathcal{V}$ has (trivially) also p-determined tolerances.

Conversely, let $\mathcal{V}$ has p-determined tolerances. Since every congruence on $\mathcal{A} \in \mathcal{V}$ is a tolerance on $\mathcal{A}$, $\mathcal{V}$ has also p-determined congruences. Moreover,

$\omega_A \in Tol\mathcal{A}$, thus we can easily derive $p(x,x) = p(y,y)$, the identity of $\mathcal{V}$. Consider again $\mathcal{A} = F_{\mathcal{V}}(x,y)$, the free algebra of $\mathcal{V}$ with two free generators x, y. By the definition of p-determined tolerances and the foregoing identity, it holds $\langle x,y\rangle \in T(p(x,y), p(x,x))$ in $\mathcal{A}$. Applying Lemma 1.7 in [3], there is a binary polynomial ψ over $\mathcal{A}$ such that $x = \psi(p(x,y), p(x,x))$ and $y = (p(x,x), p(x,y))$. Since $\mathcal{A} = F_{\mathcal{V}}(x,y)$, there is a 4-ary term q such that $\psi(z,v) = q(z,v,x,y)$.

Set $m(x,y,z) = q(p(x,y), p(y,z), x, z)$. In account of the foregoing equalities, we obtain

$$m(x,x,y) = q(p(x,x), p(x,y), x, y) = y$$

$$m(x,y,y) = q(p(x,y), p(y,y), x, y) = q(p(x,y), p(x,x), x, y) = x$$

thus $m(x,y,z)$ is a Mal'cev term which yields permutability of $\mathcal{V}$. ¶

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SOLID VARIIETES OF PARTIAL ALGEBRAS

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1 Introduction

An identity $t \approx t'$ is called a *hyperidentity* in a variety V of total algebras if whenever the operation symbols occurring in t and t' are replaced by any terms of the appropriate arity, the identity which results holds in V. As described in detail in [Den-L-P-S; 91], [Den-L-P-S; 93], and [Den-Rei; 94] for an exact definition of this concept we need a function σ, called *hypersubstitution*, which maps operation symbols to terms of the same arity. In [Den-Rei; 95] on the basis of hypersubstitutions Denecke and Reichel defined two operators χ^{E} and χ^{A}, the first one by

$$\chi^{E}[t \approx t'] := \{\hat{\sigma}[t] \approx \hat{\sigma}[t'] \mid \hat{\sigma} \text{ is the extension of a hypersubstitution}\}$$

and

$$\chi^{E}[\Sigma] = \bigcup_{t\approx t' \in \Sigma} \chi^{E}[t \approx t'].$$

for a set Σ of equations.
If $\sigma(f_i)^{A}$ is the term operation of the algebra $\underline{A} = (A; (f_i^{A})_{i\in I})$ induced by the term $\sigma(f_i)$ then $\sigma(\underline{A}) := (A; (\sigma(f_i)^{A})_{i\in I})$ is an algebra of the same type as $\underline{A}$. Now we define our second operator by

$$\chi^{A}[\underline{A}] := \{\sigma(\underline{A}) \mid \sigma \text{ is a hypersubstitution}\}$$

and

$$\chi^A[K] = \bigcup_{A \in K} \chi^A[A].$$

for a class K of algebras of the same type.

It is easy to see ([Den-Rei; 95]) that χ^A and χ^E are closure operators defined on classes of algebras of the same type and on sets of equations, respectively. Because of their definitions as unions Denecke and Reichel called them *additive* in [Den-Rei; 95]. The closure operators χ^A and χ^E are connected by the condition

$$t \approx t' \in Id\,\chi^A[A] \Leftrightarrow \chi^E[t \approx t'] \subseteq IdA. \qquad (1)$$

Here IdA is the set of all identities valid in A.
Any pair of additive closure operators with this property is called *conjugate*. If a variety V is closed with respect to the operator χ^A $(\chi^A[V] = V)$ then it is called *solid* variety. A variety is solid iff every of its identities is a hyperidentity ([Den-Rei; 95], [Den-L-P-S; 91]).
For the benefit of readers not familiar with the paper [Den-Rei; 95] we give a brief review of the main results of this paper.

1.1 Definition. Let γ_1 and γ_2 be closure operators defined on sets A and B, respectively. Then γ_1 and γ_2 are called *conjugate* to each other with respect to a relation $R \subseteq A \times B$ if the following is satisfied:
$\forall t \in A \forall s \in B(\gamma_1(\{t\}) \times \{s\} \subseteq R \Leftrightarrow \{t\} \times \gamma_2(\{s\}) \subseteq R)$.
If $\gamma_1(T) = \bigcup_{a \in T} \gamma_1(\{a\})$ and $\gamma_2(S) = \bigcup_{b \in S} \gamma_2(\{b\})$ then $\gamma := (\gamma_1, \gamma_2)$ is called a *conjugate pair of additive closure operators* (with respect to R).

If $\gamma := (\gamma_1, \gamma_2)$ is a conjugate pair of additive closure operators with respect to the relation $R \subseteq A \times B$ we define a second relation R_γ by
$R_\gamma := \{(t, s) \in A \times B \mid \gamma_1(\{t\}) \times \{s\} \subseteq R\}$

or, equivalently,
$R_\gamma := \{(t, s) \in A \times B \mid \{\{t\}\} \times \gamma_2(\{s\}) \subseteq R\}$.
The relation R_γ defines a *Galois-connection* by
$\mu_\gamma := \{s \in B \mid \forall t \in A((t, s) \in R_\gamma)\}$ and
$\iota_\gamma := \{t \in A \mid \forall s \in B((t, s) \in R_\gamma)\}$.
Assume that μ and γ are the closure operators defined in the same way by the relation R.
In the case of the closure operators χ^A and χ^B we set instead of A the class $Alg(\tau)$ of all algebras of type τ and instead of B the set $W_\tau(X) \times W_\tau(X)$ of all equations of thge same type τ. Then let

$$R := \{(A, s \approx t) \mid A \models s \approx t\}$$

be the relation defined by "the identity $s \approx t$ is satisfied in $A \in Alg(\tau)$". Because of

$$\chi^A[A] \models s \approx t \Leftrightarrow A \models \chi^E[s \approx t]$$

the pair (χ^A, χ^E) is a *conjugate pair of additive closure operators* with respect to R. Clearly, the corresponding relation R_γ is defined by

$$\begin{aligned} R_\gamma : &= \{(A, s \approx t) \in Alg(\tau) \times (W_\tau(X) \times W_\tau(X)) \mid \chi^A[A] \models s \approx t\} \\ &= \{(A, s \approx t) \in Alg(\tau) \times (W_\tau(X) \times W_\tau(X)) \mid A \models \chi^E[s \approx t]\} \end{aligned}$$

Further, for a class $K \subseteq Alg(\tau)$ of algebras of type τ and for a set $\Sigma \subseteq W_\tau(X) \times W_\tau(X)$ of equations of this type we define

$$\begin{aligned} \mu_\gamma(K) &:= \{s \approx t \in W_\tau(X) \times W_\tau(X) \mid \forall A \in K(\chi^A[A] \models s \approx t\} \text{and} \\ \iota_\gamma(\Sigma) &:= \{A \in Alg(\tau) \mid \forall s \approx t \in \Sigma(\chi^A[A] \models s \approx t\} \end{aligned}$$

Note that μ and ι are the operators assigning to every class $K \subseteq Alg(\tau)$ the set IdK of all identities valid in K and to every set Σ of equations the class $Mod\Sigma$ of all algebras satisfying the identities from Σ.

1.2 Main Theorem ([Den-Rei; 95]): *Let $\gamma = (\gamma_1, \gamma_2)$ be a conjugate pair of additive closure operators with respect to a relation $R \subseteq A \times B$. Let (μ, ι)*

denote the Galois-connection induced by R. Then for all $T \subseteq A$ *with* $\iota\mu(T) =$ T *and all* $S \subseteq B$ *with* $\mu\iota(S)$ *the following propositions* (i)–(iv) *and* (i')–(iv'), *respectively, are equivalent.*

$$
\begin{array}{llll}
(i) & T = \iota_\gamma\mu_\gamma(T) & (i') & S = \mu_\gamma\iota_\gamma(S) \\
(ii) & \gamma_1(T) = T & (ii') & \gamma_2(S) = S \\
(iii) & \mu(T) = \mu_\gamma(T) & (iii') & \iota(S) = \iota_\gamma(S) \\
(iv) & \gamma_2\mu(T) = \mu(T) & (iv') & \gamma_1\iota(S) = \iota(S)
\end{array}
$$

Moreover, we need the following results:

1.3 Theorem ([Den-Rei; 95]):*Let* $\gamma = (\gamma_1, \gamma_2)$ *be a conjugate pair of additive closure operators with respect to a relation* $R \subseteq A \times B$. *Let* (μ, ι) *denote the Galois-connection induced by R. Then for all* $T \subseteq A$ *and all* $S \subseteq B$ *the following is true.*

$$
\begin{array}{ll}
(i) & \gamma_1(T) \subseteq \iota\mu(T) \iff \iota\mu(T) = \iota_\gamma\mu_\gamma(T) \\
(i') & \gamma_2(S) \subseteq \mu\iota(S) \iff \mu\iota(S) = \mu_\gamma\iota_\gamma(S)
\end{array}
$$

It is a well-known fact that for a Galois-connection (μ, ι) between sets A and B $\mu\iota$ and $\iota\mu$ are closure operators and that the sets of all closures $\mathcal{H}_{\mu\iota}$ and $\mathcal{H}_{\iota\mu}$ are lattices with respect to inclusion. But we have even

1.4 Proposition:*Let* $\gamma = (\gamma_1, \gamma_2)$ *be a conjugate pair of additive closure operators with respect to a relation* $R \subseteq A \times B$. *Let* (μ, ι) *denote the Galois-connection induced by R and let* $(\mu_\gamma, \iota_\gamma)$ *be the Galois-connection defined by the relation* R_γ. *Then* $\mathcal{H}_{\mu_\gamma\iota_\gamma}$ *is a complete sublattice of* $\mathcal{H}_{\mu\iota}$ *and* $\mathcal{H}_{\iota_\gamma\mu_\gamma}$ *is a complete sublattice of* $\mathcal{H}_{\iota\mu}$.

In this paper we turn to partial algebras. Our aim is to give a definition of a hyperidentity for partial algebras which us allows to apply the Main Theorem 1.2 and Theorem 1.3.

2 Terms and term operations of a partial algebra

If we define terms for partial algebras $\underline{A} = (A; (f_i^A)_{i\in I})$ of type $\tau = (n_i)_{i\in I}$ where f_i^A is an n_i-ary partial operation defined on A in the usual way, then the set of all partial operations induced by terms over $\underline{A}$ is in general different from the set of all partial operations which can be produced by superposition starting with the fundamental operations of $\underline{A}$ and the projections ([Bõr; 96], [Den; 94]). So, we need a new term definition.

Let X be an alphabet and let $\{f_i | i \in I\}$ be a set of operation symbols ($X \cap \{f_i | i \in I\} = \emptyset$) of type $\tau = \{n_i | i \in I\}$ where every f_i is n_i-ary. Further we need additional symbols $\varepsilon_j^k \notin X$ for every $k \in N^* (:= N \setminus \{0\})$ and every j with $1 \le j \le k$.

Let $X_n = \{x_1, \ldots, x_n\}$ be an n-element alphabet. The set of n-ary term of type τ over X_n is defined inductively as follows (see[Bõr; 96]):

(i) every $x_i \in X_n$ is an n-ary term of type τ,

(ii) if $t_1, \ldots, t_k$ are n-ary terms then $\varepsilon_j^k(t_1, \ldots, t_k)$ is an n-ary term of type τ for all $1 \le j \le k$ and all $k \in N^*$,

(iii) if $t_1, \ldots, t_{n_i}$ are n-ary terms and if f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term of type τ.

Let $W_\tau^c(X_n)$ be the set of all n-ary terms of type τ. Then $W_\tau^c(X) := \bigcup_{n\in N^*} W_\tau^c(X_n)$ denote the set of all terms of this type.

Every n-ary term $t \in W_\tau^c(X_n)$ induces an n-ary term operation t^A of any partial algebra $\underline{A} = (A; (f_i^A)_{i\in I})$ of type τ. For $a_1, \ldots, a_n$ the value $t^A(a_1, \ldots, a_n)$

is defined in the following inductive way ([Bör; 96]):

(i) If $t = x_i$ then $t^{\underline{A}} = x_i^{\underline{A}} = e_i^{n,A}$, where $e_i^{n,A}$ is the n-ary total projection on the i-th component. So in this case $t^{\underline{A}}(a_1, \ldots, a_n)$ is defined for all $a_1, \ldots, a_n \in A$.

(ii) If $t = \varepsilon_j^k(t_1, \ldots, t_k)$ and assume that $t_1^{\underline{A}}, \ldots, t_k^{\underline{A}}$ are the term operations induced by the terms $t_1, \ldots, t_k$ and that $t_i^{\underline{A}}(a_1, \ldots, a_n), 1 \leq i \leq k$, are defined, then $t^{\underline{A}}(a_1, \ldots, a_n)$ is defined and $t^{\underline{A}}(a_1, \ldots, a_n) = t_j^{\underline{A}}(a_1, \ldots, a_n)$.

(iii) Now assume that $t = f_i(t_1, \ldots, t_{n_i})$ where f_i is an n_i-ary operation symbol and assume also that $t_1^{\underline{A}}, \ldots, t_{n_i}^{\underline{A}}$ are the term operations induced by the terms $t_1, \ldots, t_{n_i}$, and that $t_j^{\underline{A}}(a_1, \ldots, a_n), 1 \leq j \leq n_i$, are defined with values $t_1^{\underline{A}}(a_1, \ldots, a_n) = b_1, \ldots, t_{n_i}^{\underline{A}}(a_1, \ldots, a_n) = b_{n_i}$. If $f_i^{\underline{A}}(b_1, \ldots, b_{n_i})$ is defined, then $t^{\underline{A}}(a_1, \ldots, a_n)$ is defined and $t^{\underline{A}}(a_1, \ldots, a_n) = f_i^{\underline{A}}(t_1^{\underline{A}}(a_1, \ldots, a_n), \ldots, t_{n_i}^{\underline{A}}(a_1, \ldots, a_n))$.

Note that $t^{\underline{A}}$ can be the nowhere defined operation.

3 Strong identities of partial algebras

There are several possibilities to define identities in partial algebras (see e.g. [Bur; 86], [Craig; 89], [Wojd; 75]). We prefer the strong validity of equations also called *Kleene equality* (see e.g. [Rob; 89], [Sta-Sta; 89], [De; 90], [Bör; 96]) since strong identities of partial algebras are closely connected with clones of partial operations.

3.1 Definition: Let t, t' be terms of type τ $(t, t' \in W_\tau^c(X))$. Then $t \approx t'$ is called *strong identity* of the partial algebra $\underline{A} = (A; (f_i^{\underline{A}})_{i \in I})$ of type τ if $t^{\underline{A}} = t'^{\underline{A}}$ for the induced term operations (*Kleene identity*).

Our definition means that an equality $t \approx t'$ will be regarded as holding in a partial algebra iff whenever one of the term operations induced by t, t' is defined so is the other, and, when both are defined, then they have the same value. In this case we write $\underline{A} \models_s t \approx t'$. Let $Id_s\underline{A}$ be the class of all strong identities of $\underline{A}$ and let $PAlg(\tau)$ be the class of all partial algebras of type τ. If K is a subclass of $PAlg(\tau)$ and if $t \approx t'$ is a strong identity in every algebra of K we write $K \models_s t \approx t'$. Let Id_sK be the class of all strong identities of K. For a set $\Sigma \subseteq W_\tau^c(X) \times W_\tau^c(X)$ we define

$$Mod_s\Sigma = \{\underline{A} \mid \underline{A} \in PAlg(\tau) \text{ and } \underline{A} \models_s t \approx t' \text{ for every equation } t \approx t' \in \Sigma, \text{ i.e. } \underline{A} \models_s \Sigma\}.$$

The class $Mod_s\Sigma$ is called *strong equational class* or *strong variety* defined by Σ. Clearly, $\models_s$ defines a relation between $PAlg(\tau)$ and $W_\tau^c(X) \times W_\tau^c(X)$. Note that F. Börner in [Bör; 96] gave a Birkhoff-type-characterization of strong equational classes of partial algebras as classes closed under *subalgebras*, *closed homomorphic images* and *filtered products*. Moreover, he proved that $V \subseteq PAlg(\tau)$ is a strong variety of partial algebras iff $V = Mod_sId_sV$.

4 Term clone and clone of term operations of a partial algebra

Assume that $t_1, \ldots, t_m$ are n-ary terms and that t is m-ary. Then we define an n-ary term $\overline{S}_n^m(t, t_1, \ldots, t_m)$ inductively by the following steps:

(i) For $t = x_j$, $1 \leq j \leq m$ (m-ary variable), we define $\overline{S}_n^m(x_j, t_1, \ldots, t_m) = t_j$.

(ii) For $t = \varepsilon_j^k(s_1, \ldots, s_k)$ we set
$\overline{S}_n^m(t, t_1, \ldots, t_m) = \varepsilon_j^k(\overline{S}_n^m(s_1, t_1, \ldots, t_m), \ldots, \overline{S}_n^m(s_m, t_1, \ldots, t_m))$, where $s_1, \ldots, s_k$ are m-ary, for all $k \in \mathbb{N}^*$ and $1 \leq j \leq k$.

(iii) For $t = f_i(s_1, \ldots, s_{n_i})$ we set
$\overline{S}_n^m(t, t_1, \ldots, t_m) = f_i(\overline{S}_n^m(s_1, t_1, \ldots, t_m), \ldots, \overline{S}_n^m(s_k, t_1, \ldots, t_m))$, where $s_1, \ldots, s_{n_i}$ are again m-ary.

This defines an operation

$$\overline{S}^m_n : W^c_\tau(X_m) \times (W^c_\tau(X_n))^m \to W^c_\tau(X_n) ,$$

which describes the superposition of terms.

The *term clone* of type τ is the heterogeneous algebra

$\underline{\text{clone}\,\tau} := ((W^c_\tau(X_n))_{n\in \mathbb{N}^*}; (\overline{S}^m_n)_{n,m\in \mathbb{N}^*}, (e^k_j)_{k\in \mathbb{N}^*, 1\le j\le k})$ with $e^k_j = x_j \in X_k, 1 \le j \le k$.

Superposition can also be defined for partial operations $f : A^n \multimap A$. Let $P^{(n)}(A)$ be the set of all n-ary partial operations and let $P^{(m)}(A)$ be the set of all m-ary partial operations. Then we define operations $S^n_m : P^{(n)}(A) \times (P^{(m)}(A))^n \to P^{(m)}(A)$ by

$$S^n_m(f^A, g^A_1, \ldots, g^A_n)(a_1, \ldots, a_m) := f^A(g^A_1(a_1, \ldots, a_m), \ldots, g^A_n(a_1, \ldots, a_m))$$

for all $(a_1, \ldots, a_m)$ for which $g^A_1, \ldots, g^A_n$ are defined and for which the values $b_1 = g^A_1(a_1, \ldots, a_m), \ldots, b_n = g^A_n(a_1, \ldots, a_m)$ form an n-tuple $(b_1, \ldots, b_n)$ belonging to the domain of f^A.

Further, let $e^{n,A}_i$ be the n-ary projection on the i-th component.
Every subalgebra of the heterogeneous algebra

$$((P^{(n)}(A))_{n\in \mathbb{N}^*}, (S^n_m)_{n,m\in \mathbb{N}^*}, (e^{n,A}_i)_{n\in \mathbb{N}^*, 1\le i\le n})$$

is called *partial clone*. This concept was used by A.I. Mal'cev ([Mal; 74]) (see also [Hoe; 77]).

All partial clones are in a variety $\mathbb{K}_1$ which was equationally described in [Bör; 96], and every element of this variety is isomorphic to a subdirect product of partial clones. The term clone is also contained in this variety, but it is an element of a proper subvariety $\mathbb{K}_0$ of $\mathbb{K}_1$. An axiom system for $\mathbb{K}_0$ is given in several papers, see e.g. [Tay; 73].

Let $T(\underline{A})^{(n)}$ be the set of all n-ary term operations of a partial algebra $\underline{A} = (A; (f^{\underline{A}}_i)_{i\in I})$. Then $T(\underline{A}) = ((T(\underline{A})^{(n)})_{n\in \mathbb{N}^*}, (S^n_m)_{m,n\in \mathbb{N}^*}, (e^{n,A}_i)_{n\in \mathbb{N}^*, 1\le i\le n})$ is also a partial clone (*clone of term operations* of $\underline{A}$), namely the partial clone $\langle (F^{(n)}(\underline{A}))_{n\in \mathbb{N}^*}\rangle$ with $F^{(n)}(\underline{A}) = \{f^{\underline{A}}_i \mid i \in I\}^{(n)}$ ([Bör; 93]), i.e. the

partial clone generated by the fundamental operations of the algebra $\underline{A} = (A; (f^{\underline{A}}_i)_{i\in I})$.

There is a close interconnection between $\underline{\text{clone}\,\tau}$ and $T(\underline{A})$.

4.1 Theorem. *Let* $\underline{\text{clone}\,\tau} = ((W^c_\tau(X_n))_{n\in \mathbb{N}^*}, (\overline{S}^m_n)_{m,n\in \mathbb{N}^*}, (e^k_j)_{k\in \mathbb{N}^*, 1\le j\le k})$ *be the term clone of type* τ, *let* $\underline{A} = (A; (f^{\underline{A}}_i)_{i\in I})$ *be an algebra of type* τ *and let* $T(\underline{A}) = ((T(\underline{A})^{(n)})_{n\in \mathbb{N}^*}, (S^n_m)_{m,n\in \mathbb{N}^*}, (e^{n,A}_i)_{n\in \mathbb{N}^*, 1\le i\le n})$ *be the clone of term operations of* $\underline{A}$.
Then there is a family $\varphi = (\varphi^{(n)})_{n\in \mathbb{N}^*}$ *of mappings,* $\varphi^{(n)} : W^c_\tau(X_n) \to T(\underline{A})^{(n)}$ *which fulfils the following properties*

(i) $\varphi^{(n)}(e^n_i) = e^{n,A}_i$, $1 \le i \le n$, $n \in \mathbb{N}^*$,

(ii) *for* $t \in W^c_\tau(X_n)$ *we define* $\varphi^{(n)}(t) = t^{\underline{A}}$ *where* $t^{\underline{A}}$ *is the term operation induced by* t *which is also* n*-ary. Then*
$\varphi^{(n)}(\overline{S}^m_n(s, t_1, \ldots, t_m))\big|_D = S^m_n(\varphi^{(m)}(s), \varphi^{(n)}(t_1), \ldots, \varphi^{(n)}(t_m))\big|_D$, $n \in \mathbb{N}^*$,
where D *is the intersection of the domains of all* $t^{\underline{A}}_i$, $1 \le i \le m$, *where* s *is* m*-ary, and* $t_1, \ldots, t_m$ *are* n*-ary.*

Proof. By definition, $\varphi^{(n)}$ is a function for every $n \in \mathbb{N}^*$. Then we have

(i) : $\varphi^{(n)}(e^n_i) = e^{n,A}_i$ for every $n \in \mathbb{N}^*$ and $1 \le i \le n$.

(ii) : For the proof of the second proposition we use induction with respect to the complexity of the definition of s. If $s = x_i \in X_m$ then

$$\varphi^{(n)}(\overline{S}^m_n(x_i, t_1, \ldots, t_m))\big|_D = t^{\underline{A}}_i\big|_D = S^m_n(\varphi^{(n)}(x_i), \varphi^{(n)}(t_1), \ldots, \varphi^{(n)}(t_m))\big|_D .$$

If $s = \varepsilon^k_j(s_1, \ldots, s_k)$ and assume that for $s_1, \ldots, s_k \in W^c_\tau(X_m)$ the condition (ii) is satisfied then

$$\begin{aligned}
&\varphi^{(n)}(\overline{S}^m_n(\varepsilon^k_j(s_1, \ldots, s_k), t_1, \ldots, t_m))\big|_D \\
&= \varphi^{(n)}(\varepsilon^k_j(\overline{S}^m_n(s_1, t_1, \ldots, t_m), \ldots, \overline{S}^{(m)}_n(s_k, t_1, \ldots, t_m)))\big|_D \\
&= S^k_n(e^{k,A}_j, \varphi^{(n)}(\overline{S}^m_n(s_1, t_1, \ldots, t_m))\big|_D, \ldots, \varphi^{(n)}(\overline{S}^m_n(s_k, t_1, \ldots, t_m))
\end{aligned}$$

$$\begin{aligned} &= S_n^k(e_j^{k,\mathcal{A}}, \overline{S}_n^m(s_1^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_m^{\mathcal{A}})\Big|_D, \ldots, \overline{S}_n^m(s_k^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_m^{\mathcal{A}})\Big|_D) \\ &= S_n^m(\varepsilon_j^k(s_1, \ldots, s_k)^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_m^{\mathcal{A}})\Big|_D \\ &= S_n^m(\varphi^{(n)}(\varepsilon_j^k(s_1, \ldots, s_k)), \varphi^{(n)}(t_1), \ldots, \varphi^{(n)}(t_m)\Big|_D . \end{aligned}$$

Finally, assume that $s = f_i(s_1, \ldots, s_{n_i})$.
Then $\varphi^{(n)}(\overline{S}_n^m(s, t_1, \ldots, t_m)) = S_n^m(\varphi^{(m)}(s), \varphi^{(n)}(t_1), \ldots, \varphi^{(n)}(t_m))$ if $t_1^{\mathcal{A}}, \ldots, t_m^{\mathcal{A}}$ are defined and for all $(a_1, \ldots, a_n) \in D$ the m-tuple $(b_1, \ldots, b_m)$ with $t_1^{\mathcal{A}}(a_1, \ldots, a_n) = b_1, \ldots, t_m^{\mathcal{A}}(a_1, \ldots, a_n) = b_m$ is in the domain of $s^{\mathcal{A}}$. If $t_1^{\mathcal{A}}, \ldots, t_m^{\mathcal{A}}$ are defined and there is an $(a_1, \ldots, a_n) \in D$ for which $(b_1, \ldots, b_m)$ is not in the domain of $s^{\mathcal{A}}$ then both sides are not defined. That means, in any case we have
$\varphi^{(n)}(\overline{S}_n^m(s, t_1, \ldots, t_m))\Big|_D = S_n^m(\varphi^{(m)}(s), \varphi^{(n)}(t_1), \ldots, \varphi^{(n)}(t_m))\Big|_D$ and this finishes the proof. ∎

5 Hypersubstitutions of type τ

Our new definition of terms in the partial case means that we also have to modify slightly our definition of a hypersubstitution.

5.1 Definition. Let $\{f_i \mid i \in I\}$ be a set of operation symbols of type τ and let $W_\tau^c(X)$ be the set of all terms of this type.
A mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau^c(X)$ which preserves the arity is called *hypersubstitution of type* τ.

Any hypersubstitution of type τ can be extended to a map defined on all terms in the following inductive way. Here $\hat{\sigma} : W_\tau^c(X) \to W_\tau^c(X)$ is the extension of the hypersubstitution σ.

(i) $\hat{\sigma}[x_i] = x_i$ for every $x_i \in X_n$,

(ii) $\hat{\sigma}[\varepsilon_j^k(s_1, \ldots, s_k)] = \overline{S}_n^k(\varepsilon_j^k(x_1, \ldots, x_k), \ \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_k]), \quad s_1, \ldots, s_k \in W_\tau^c(X_n)$,

(iii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] = \overline{S}_n^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]), \quad t_1, \ldots, t_{n_i} \in W_\tau^c(X_n)$.

Since the extension $\hat{\sigma}$ of a hypersubstitution σ preserves arities, every extension $\hat{\sigma}$ defines a family of mappings $\hat{\sigma} := (\eta^{(n)} : W_\tau^c(X_n) \to W_\tau^c(X_n))_{n \in \mathbb{N}}$. We prove

5.2 Theorem. *The extension $\hat{\sigma}$ of a hypersubstitution σ of type τ defines an endomorphism $(\eta^{(n)})_{n \in \mathbb{N}}$ of* $\underline{clone\ \tau}$.

Proof. At first we prove that $(\eta^{(n)})_{n \in \mathbb{N}}$ preserves the constant operations e_j^k. Indeed, $\eta^{(k)}(e_j^k) = \eta^{(k)}(x_j) = \hat{\sigma}[x_j] = x_j = e_j^k$.
Now we are going to prove the compatibility with the operations $\overline{S}_n^m$ by induction on the complexity of the term definition of $s \in W_\tau^c(X_m)$ in $\overline{S}_n^m(s, t_1, \ldots, t_m)$, $t_1, \ldots, t_m \in W_\tau^c(X_n)$. If s an m-ary variable x_j, $1 \le j \le m$, then we have

$$\begin{aligned} \eta^{(n)}(\overline{S}_n^m(x_j, t_1, \ldots, t_m)) &= \hat{\sigma}[t_j] = \overline{S}_n^m(\hat{\sigma}[x_j], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_m]) \\ &= \overline{S}_n^m(\eta^{(m)}(x_j), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)) . \end{aligned}$$

If $s = \varepsilon_j^k(s_1, \ldots, s_k)$ and assume that for $s_1, \ldots, s_k \in W_\tau^c(X_m)$ the permutability condition is satisfied then

$$\begin{aligned} &\eta^{(n)}(\overline{S}_n^m(\varepsilon_j^k(s_1, \ldots, s_k), t_1, \ldots, t_m)) = \hat{\sigma}[\overline{S}_n^m(\varepsilon_j^k(s_1, \ldots, s_k), t_1, \ldots, t_m)] \\ &= \hat{\sigma}[\varepsilon_j^k(\overline{S}_n^m(s_1, t_1, \ldots, t_m), \ldots, \overline{S}_n^m(s_k, t_1, \ldots, t_m))] \\ &= \overline{S}_n^k(\varepsilon_j^k(x_1, \ldots, x_k), \hat{\sigma}[\overline{S}_n^m(s_1, t_1, \ldots, t_m)], \ldots, \hat{\sigma}[\overline{S}_n^m(s_k, t_1, \ldots, t_m)]) \\ &= \overline{S}_n^k(\varepsilon_j^k(x_1, \ldots, x_k), \overline{S}_n^m(\eta^{(m)}(s_1), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)), \ldots, \\ &\qquad \overline{S}_n^m(\eta^{(m)}(s_k), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m))) \\ &= \varepsilon_j^k(\overline{S}_n^k(x_1, \overline{S}_n^m(\eta^{(m)}(s_1), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)), \ldots, \\ &\qquad \overline{S}_n^m(\eta^{(m)}(s_k), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m))), \ldots, \\ &\qquad \overline{S}_n^k(x_k, \overline{S}_n^m(\eta^{(m)}(s_1), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)), \ldots, \\ &\qquad \overline{S}_n^m(\eta^{(m)}(s_k), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)))) \\ &= \varepsilon_j^k(\overline{S}_n^m(\eta^{(m)}(s_1), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)), \ldots, \overline{S}_n^m(\eta^{(m)}(s_k), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m))) \\ &= \overline{S}_n^m(\varepsilon_j^k(\eta^{(m)}(s_1), \ldots, \eta^{(m)}(s_k)), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)). \end{aligned}$$

Now assume that $s = f_i(s_1, \ldots, s_{n_i})$ and further we assume that for $s_1, \ldots, s_{n_i} \in W_\tau^c(X_m)$ the permutability condition is satisfied. Then

$$\begin{aligned}
\eta^{(n)}(\overline{S}_n^m(s, t_1, \ldots, t_m)) &= \eta^{(n)}(\overline{S}_n^m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_m)) \\
&= \hat{\sigma}[f_i(\overline{S}_n^m(s_1, t_1, \ldots, t_m), \ldots, \overline{S}_n^m(s_{n_i}, t_1, \ldots, t_m))] \\
&= \overline{S}_n^{n_i}(\sigma(f_i), \hat{\sigma}[\overline{S}_n^m(s_1, t_1, \ldots, t_m)], \ldots, \hat{\sigma}[\overline{S}_n^m(s_{n_i}, t_1, \ldots, t_m)]) \\
&= \overline{S}_n^{n_i}(\eta^{(n_i)}(f_i(x_1, \ldots, x_{n_i})), \overline{S}_n^m(\eta^{(m)}(s_1), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)), \ldots \\
&\quad \overline{S}_n^m(\eta^{(m)}(s_{n_i}), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m))) \\
&= \overline{S}_n^m(\eta^{(m)}(f_i(s_1, \ldots, s_{n_i})), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)) \\
&= \overline{S}_n^m(\eta^{(m)}(s), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)).
\end{aligned}$$

Hypersubstitutions can be applied to strong identities of partial algebras $\underline{A}$ and we are asking whether the resulting equations $\hat{\sigma}[t] \approx \hat{\sigma}[t']$ are again strong identities of $\underline{A}$. Hypersubstitutions can also be applied to partial algebras of type τ and we obtain partial algebras $\sigma[\underline{A}] := (A; (\sigma(f_i)^{\underline{A}})_{i \in I})$. Based on these definitions one defines two closure operators χ^E and χ^A. As mentioned in the introduction in the total case this is a conjugate pair of additive closure operators what has farreaching consequences. In the partial case this is no longer true since in general the following condition is not satisfied

$$\underline{A} \models_s \hat{\sigma}[t] \approx \hat{\sigma}[t'] \Leftrightarrow \sigma[\underline{A}] \models_s t \approx t' \qquad (C)$$

as the following example shows:
Let $\underline{A} = (A; f^{\underline{A}}, n^{\underline{A}})$, $|A| > 1$, be a partial algebra of type $\tau = (2,1)$ where $f^{\underline{A}}$ is an arbitrary binary operation and where $n^{\underline{A}}$ is the unary operation with empty domain.

Clearly,

$$\underline{A} \models_s f(x_1, n(x_2)) \approx f(x_2, n(x_1)). \qquad (1)$$

Consider the hypersubstitution σ defined by $\sigma(f) = x_1$, $\sigma(n) = n(x_1)$.
Then applying σ to the Algebra $\underline{A}$ we have $\sigma[\underline{A}] = (A; \sigma(f)^{\underline{A}}, \sigma(n)^{\underline{A}})$

$(A; e_1^{1,A}, n^A)$ and $\sigma[\underline{A}] \models_s (1)$ since $f(x_1, n(x_2))^{\underline{A}} = e_1^{2,\underline{A}}(e_1^{2,\underline{A}}, n^{\underline{A}})$ is nowhere defined and $f(x_2, n(x_1))^{\underline{A}} = e_1^{2,\underline{A}}(e_2^{2,\underline{A}}, n^{\underline{A}})$ is also nowhere defined.
Further we have $\hat{\sigma}[f(x_1, n(x_2))] = \overline{S}_2^2(\sigma(f), x_1, n(x_2)) = x_1$ and $\hat{\sigma}[f(x_2, n(x_1))] = \overline{S}_2^2(\sigma(f), x_2, n(x_1)) = x_2$. This shows $\underline{A} \not\models_s \hat{\sigma}[f(x_1, n(x_2))] \approx \hat{\sigma}[f(x_2, n(x_1))]$ since $|A| > 1$.

It turns out that the proper reason for the non-satisfaction of condition (C) is that the set of variables occuring in $\hat{\sigma}[f(x_1, x_2)]$ is different from $\{x_1, x_2\}$. So, we have to consider so-called regular hypersubstitutions.

Regular hypersubstitutions

By $Var(\hat{\sigma}[f_i(x_1, \ldots, x_{n_i})])$ we denote the set of all variables occuring in $\hat{\sigma}[f_i(x_1, \ldots, x_{n_i})]$.

Definition. A hypersubstitution $\sigma : \{f_i \mid i \in I\} \to W_\tau^c(X)$ is called regular hypersubstitution if $Var(\hat{\sigma}[f_i(x_1, \ldots, x_{n_i})]) = \{x_1, \ldots, x_{n_i}\}$.

The next step is to show that the set $Hyp_R^c(\tau)$ of all R-hypersubstitutions of type τ forms a monoid. To check this we need the following lemma:

Lemma. *For any two R-hypersubstitutions σ_1, σ_2 of type τ we have*

$$(\hat{\sigma}_2 \circ \sigma_1)^{\hat{}} = \hat{\sigma}_2 \circ \hat{\sigma}_1 ,$$

where $\circ$ is the usual composition of functions.

Proof. $\hat{\sigma}_2 \circ \sigma_1$ maps operation symbols of type τ to terms of this type by

$$\{f_i \mid i \in I\} \xrightarrow{\sigma_1} W_\tau^c(X) \xrightarrow{\hat{\sigma}_2} W_\tau^c(X).$$

Since arities are preserved, the product $\hat{\sigma}_2 \circ \sigma_1$ is a hypersubstitution of type τ. Using induction we are going to show the equality $(\hat{\sigma}_2 \circ \sigma_1)^{\hat{}}[t] = (\hat{\sigma}_2 \circ \hat{\sigma}_1)[t]$ for any term t.

Assume that $x_i \in X$ is a variable then $(\hat{\sigma}_2 \circ \sigma_1)^{\hat{}}[x_i] = x_i = (\hat{\sigma}_2 \circ \hat{\sigma}_1)[x_i] = \hat{\sigma}_2[\hat{\sigma}_1[x_i]]$. If $t = e_j^k(s_1, \ldots, s_k)$, $1 \leq j \leq k$, then

$(\hat{\sigma}_2 \circ \sigma_1)^{\hat{}}[\epsilon_j^k(s_1,\ldots,s_k)] = \overline{S}_n^k(\epsilon_j^k(x_1,\ldots,x_k),(\hat{\sigma}_2 \circ \sigma_1)^{\hat{}}[s_1],\ldots,(\hat{\sigma}_2 \circ \sigma_1)^{\hat{}}[s_k])$
$= \epsilon_j^k(\overline{S}_n^k(x_1,(\hat{\sigma}_2 \circ \sigma_1)^{\hat{}}[s_1],\ldots,(\hat{\sigma}_2 \circ \sigma_1)^{\hat{}}[s_k]),\ldots,\overline{S}_n^k(x_k,(\hat{\sigma}_2 \circ \sigma_1)^{\hat{}}[s_1],\ldots,(\hat{\sigma}_2 \circ$
$\sigma_1)^{\hat{}}[s_k]))$
$= \epsilon_j^k(\overline{S}_n^k(x_1,(\hat{\sigma}_2 \circ \hat{\sigma}_1)[s_1],\ldots,(\hat{\sigma}_2 \circ \hat{\sigma}_1)[s_k]),\ldots,\overline{S}_n^k(x_k,(\hat{\sigma}_2 \circ \hat{\sigma}_1)[s_1],\ldots,(\hat{\sigma}_2 \circ$
$\hat{\sigma}_1)[s_k]))$
$= \epsilon_j^k((\hat{\sigma}_2 \circ \hat{\sigma}_1)[s_1],\ldots,(\hat{\sigma}_2 \circ \hat{\sigma}_1)[s_k])$

by induction hypothesis. On the other hand we have

$$\begin{aligned}
(\hat{\sigma}_2 \circ \hat{\sigma}_1)[\epsilon_j^k(s_1,\ldots,s_k)] &= \hat{\sigma}_2[\hat{\sigma}_1[\epsilon_j^k(s_1,\ldots,s_k)]]\\
&= \hat{\sigma}_2[\overline{S}_n^k(\epsilon_j^k(x_1,\ldots,x_k),\hat{\sigma}_1[s_1],\ldots,\hat{\sigma}_1[s_k])]\\
&= \hat{\sigma}_2[\epsilon_j^k(\overline{S}_n^k(x_1,\hat{\sigma}_1[s_1],\ldots,\hat{\sigma}_1[s_k]),\ldots,\overline{S}_n^k(x_k,\hat{\sigma}_1[s_1],\ldots,\hat{\sigma}_1[s_k]))]\\
&= \hat{\sigma}_2[\epsilon_j^k(\hat{\sigma}_1[s_1],\ldots,\hat{\sigma}_1[s_k])]\\
&= \overline{S}_n^k(\epsilon_j^k(x_1,\ldots,x_k),\hat{\sigma}_2[\hat{\sigma}_1[s_1]],\ldots,\hat{\sigma}_2[\hat{\sigma}_1[s_k]])\\
&= \epsilon_j^k((\hat{\sigma}_2 \circ \hat{\sigma}_1)[s_1],\ldots,(\hat{\sigma}_2 \circ \hat{\sigma}_1)[s_k]).
\end{aligned}$$

So, we get $(\hat{\sigma}_2 \circ \sigma_1)^{\hat{}}[t] = (\hat{\sigma}_2 \circ \hat{\sigma}_1)[t]$.
If $t = f_i(s_1,\ldots,s_{n_i})$, then $(\hat{\sigma}_2 \circ \sigma_1)^{\hat{}}[f_i(s_1,\ldots,s_{n_i})] = \overline{S}_n^{n_i}((\hat{\sigma}_2 \circ \sigma_1)(f_i),(\hat{\sigma}_2 \circ$
$\hat{\sigma}_1)[s_1],\ldots,(\hat{\sigma}_2 \circ \hat{\sigma}_1)[s_{n_i}])$ by induction hypothesis and
$(\hat{\sigma}_2 \circ \hat{\sigma}_1)[f_i(s_1,\ldots,s_{n_i})] = \hat{\sigma}_2[\hat{\sigma}_1[f_i(s_1,\ldots,s_{n_i})]]$
$\hat{\sigma}_2[\overline{S}_n^{n_i}(\sigma_1(f_i),\hat{\sigma}_1[s_1],\ldots,\hat{\sigma}_1[s_{n_i}])]$
$= \overline{S}_n^{n_i}(\hat{\sigma}_2[\sigma_1(f_i)],\hat{\sigma}_2[\hat{\sigma}_1[s_1]],\ldots,\hat{\sigma}_2[\hat{\sigma}_1[s_{n_i}]])$
$= \overline{S}_n^{n_i}((\hat{\sigma}_2 \circ \sigma_1)(f_i),(\hat{\sigma}_2 \circ \hat{\sigma}_1)[s_1],\ldots,(\hat{\sigma}_2 \circ \hat{\sigma}_1)[s_{n_i}])$. Here we used Theorem 5.2

■

Now we define a product of hypersubstitutions by $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_2 \circ \sigma_1$ and obtain:

6.3 Theorem. *$(Hyp_R^c(\tau);\ \circ_h;\ \sigma_{id})$ with $\sigma_{id}(f_i) = f_i(x_1,\ldots,x_{n_i})$ is a monoid*

Proof. The identity hypersubstitution σ_{id} is obviously regular. The product of two regular hypersubstitutions σ_1,σ_2 of type τ is also regular. To prove this we show at first by induction on the complexity of term definition that for any n-ary term t and any regular hypersubstitution σ the sets $Var(t)$ and $Var(\hat{\sigma}[t])$ are equal.

If $t = x_i$ is a variable then this is clear.

Consider the term $t = \epsilon_j^k(t_1,\ldots,t_k)$ and assume that for the regular hypersubstitution σ there holds $Var(t_i) = Var(\hat{\sigma}[t_i])$, $1 \leq i \leq k$. Then $Var(\epsilon_j^k(t_1,\ldots,t_k)) = \bigcup\limits_{i=1}^{k} Var(t_i)$ and $Var(\hat{\sigma}[\epsilon_j^k(t_1,\ldots,t_k)]) = Var(\overline{S}_n^k(\epsilon_j^k(x_1,\ldots,x_k),\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_k])) = Var(\epsilon_j^k(\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_k])) = \bigcup\limits_{i=1}^{k} Var(t_i)$ because of our presumption.

If finally $t = f_i(t_1,\ldots,t_{n_i})$ and if we assume that $Var(t_j) = Var(\hat{\sigma}[t_j])$, $1 \leq j \leq n_i$, then $Var(f_i(t_1,\ldots,t_{n_i})) = \bigcup\limits_{j=1}^{n_i} Var(t_j)$ and $Var(\hat{\sigma}[f_i(t_1\ldots,t_{n_i})]) = Var(\overline{S}_n^{n_i}(\sigma(f_i),\hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_{n_i}])) = \bigcup\limits_{j=1}^{n_i} Var(\hat{\sigma}[t_j]) = \bigcup\limits_{j=1}^{n_i} Var(t_j)$.

Now for the product $\sigma_1 \circ_h \sigma_2$ of two regular hypersubstitutions σ_1,σ_2 we have $Var((\sigma_1 \circ_h \sigma_2)^{\hat{}}[f_i(x_1,\ldots,x_{n_i})]) = Var(\hat{\sigma}_2[\hat{\sigma}_1[f_i(x_1,\ldots,x_{n_i})]]) = Var(\hat{\sigma}_1[f_i(x_1,\ldots,x_{n_i})]) = Var(f_i(x_1,\ldots,x_{n_i})) = \{x_1,\ldots,x_{n_i}\}$.

The following calculation shows the associativity of our product $\circ_h$:
$(\sigma_1 \circ_h \sigma_2) \circ_h \sigma_3 = (\hat{\sigma}_2 \circ \sigma_1) \circ_h \sigma_3 = \hat{\sigma}_3 \circ (\hat{\sigma}_2 \circ \sigma_1) = (\hat{\sigma}_3 \circ \hat{\sigma}_2) \circ \sigma_1 = (\hat{\sigma}_3 \circ \sigma_2)^{\hat{}} \circ \sigma_1 = \sigma_1 \circ_h (\sigma_2 \circ_h \sigma_3)$. ■

If $\mathcal{A} = (A;(f_i^{\mathcal{A}})_{i\in I})$ is a partial algebra of type $\tau = (n_i)_{i\in I}$, then for any $\sigma \in Hyp_R^c(\tau)$ we consider the algebra $\sigma[\mathcal{A}] = (A;(\sigma(f_i)^{\mathcal{A}})_{i\in I})$ where $\sigma(f_i)^{\mathcal{A}}$ is the term operation induced by the term $\sigma(f_i)$ on the algebra $\mathcal{A}$.

For regular hypersubstitutions we prove:

6.4 Lemma. *Let σ be a regular hypersubstitution of type τ and let $\sigma[\underline{A}] = (A; (\sigma(f_i)^{\underline{A}})_{i\in I})$. For a term $t \in W_\tau^c(X)$ by $t^{\sigma[\underline{A}]}$, respectively by $\hat{\sigma}[t]^{\underline{A}}$, we denote the term operations of the algebra $\sigma[\underline{A}]$, respectively of the algebra $\underline{A}$, induced by the term t in the first and by the term $\hat{\sigma}[t]$ in the second case. Then for every term $t \in W_\tau^c(X)$ we get*

$$\hat{\sigma}[t]^{\underline{A}} = t^{\sigma[\underline{A}]}.$$

Proof. We are proving this equation using complexity of the definition of the term t. If $t = x_i$ is a variable ($x_i \in X_n$) then $\hat{\sigma}[x_i]^{\underline{A}} = x_i^{\underline{A}} = e_i^{n,A} = x_i^{\sigma[\underline{A}]}$. Assume now that $t = \varepsilon_j^k(t_1,\ldots,t_k)$ and that $\hat{\sigma}[t_i]^{\underline{A}} = t_i^{\sigma[\underline{A}]}$ for every $i \in \{1,\ldots,k\}$. Then $\hat{\sigma}[t]^{\underline{A}} = \hat{\sigma}[\varepsilon_j^k(t_1,\ldots,t_k)]^{\underline{A}} = (\overline{S}_{\tau}^k(\varepsilon_j^k(x_1,\ldots,x_k), \hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_k]))^{\underline{A}} = (\varepsilon_j^k(t_1,\ldots,t_k))^{\sigma[\underline{A}]}$ by Theorem 4.1 if both sides are defined and otherwise both sides are not defined.

If finally $t = f_i(t_1,\ldots,t_{n_i})$ and assume again that $\hat{\sigma}[t_j]^{\underline{A}} = t_j^{\sigma[\underline{A}]}$ for every $j \in \{1,\ldots,n_i\}$, then we consider the following two cases:

1. $\sigma(f_i) = x_j \in X_{n_i}$.
   Then $\hat{\sigma}[t] = \overline{S}_n^{n_i}(x_j, \hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_{n_i}]) = \hat{\sigma}[t_j]$ and therefore $\hat{\sigma}[t]^{\underline{A}} = \hat{\sigma}[t_j]^{\underline{A}}$.
   Since σ is a regular hypersubstitution we have $Var(f_i(x_1,\ldots,x_{n_i})) = x_j$.
   If follows that $n_i = j = 1$ and therefore $\hat{\sigma}[t]^{\underline{A}} = \hat{\sigma}[t_1]^{\underline{A}}$.
   Calculating $t^{\sigma[\underline{A}]}$ we get $(f_i(t_1,\ldots,t_{n_i}))^{\sigma[\underline{A}]} = t_1^{\sigma[\underline{A}]}$. By our presumption we have $\hat{\sigma}[t_1]^{\underline{A}} = t_1^{\sigma[\underline{A}]}$ and thus $t^{\sigma[\underline{A}]} = \hat{\sigma}[t]^{\underline{A}}$.

2. There is no $j \in \{1,\ldots,n_i\}$ with $\sigma(f_i) = x_j \in X_{n_i}$.
   Then
   $\hat{\sigma}[t]^{\underline{A}} = \overline{S}_n^{n_i}(\sigma(f_i), \hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_{n_i}])^{\underline{A}} = S_n^{n_i}(\sigma(f_i)^{\underline{A}}, \hat{\sigma}[t_1]^{\underline{A}},\ldots,\hat{\sigma}[t_{n_i}]^{\underline{A}})$ for every n-tuple belonging to the intersection D of all domains of $\sigma[t_j]^{\underline{A}}$, $1 \leq j \leq n_i$, by Theorem 4.1.
   Using the hypothesis one gets
   $$\hat{\sigma}[t]^{\underline{A}} = S_n^{n_i}(\sigma(f_i)^{\underline{A}}, t_1^{\sigma[\underline{A}]},\ldots,t_{n_i}^{\sigma[\underline{A}]}) = (f_i(t_1,\ldots,t_{n_i}))^{\sigma[\underline{A}]} = t^{\sigma[\underline{A}]}$$
   for all n-tuples belonging to the intersection D of all domains of $t_j^{[\underline{A}]}$, $1 \leq j \leq n_i$.

If we consider an n-tupel for which one of the operations $\hat{\sigma}[t_j]^{\underline{A}}$ is not defined then $\overline{S}_n^{n_i}(\sigma(f_i), \hat{\sigma}[t_1],\ldots,\hat{\sigma}[t_{n_i}])^{\underline{A}}$ is not defined since because of the regularity of σ for any $j \in \{1,\ldots,n_i\}$ the term $\hat{\sigma}[t_j]$ occurs in $\hat{\sigma}[t]$. The equality $\hat{\sigma}[t_j]^{\underline{A}} = t_j^{\sigma[\underline{A}]}$ for every $j \in \{1,\ldots,n_i\}$ shows that then $t^{\sigma[\underline{A}]} = (f_i(t_1,\ldots,t_{n_i}))^{\sigma[\underline{A}]}$ is also not defined. Therefore, we get $\hat{\sigma}[t]^{\underline{A}} = t^{\sigma[\underline{A}]}$. ∎

7 The closure operators χ_M^E and χ_M^A

Let $M \subseteq (Hyp_R^c(\tau); \circ_h, \sigma_{id})$ be a submonoid of the monoid of all R-hypersubstitutions of type τ. In the same manner as in [Den-Rei; 95] Denecke and Reichel introduced two operators χ_M^E, χ_M^A by
$\chi_M^E[t \approx t'] := \{\hat{\sigma}[t] \approx \hat{\sigma}[t'] \mid \hat{\sigma}$ is the extension of a hypersubstitution from $M\}$
for a pair $(t \approx t') \in W_\tau^c(X) \times W_\tau^c(X)$ and
$\chi_M^E[\Sigma] := \bigcup_{(t,t')\in\Sigma} \chi_M^E[t \approx t']$ for a set $\Sigma \subseteq W_\tau^c(X) \times W_\tau^c(X)$.

If $\underline{A} = (A; (f_i^{\underline{A}})_{i\in I})$ is a partial algebra of type $\tau = (n_i)_{i\in I}$, then for any $\sigma \in M$ we consider the algebra $\sigma[\underline{A}] = (A; (\sigma(f_i)^{\underline{A}})_{i\in I})$ as defined in section 6. Clearly, $\sigma[\underline{A}]$ has also the type τ since hypersubstitutions preserve arities. Further we define $\chi_M^A[\underline{A}] := \{\sigma[\underline{A}] \mid \sigma$ is a hypersubstitution from $M\}$ and $\chi_M^A[K] := \bigcup_{\underline{A}\in K} \chi_M^A[\underline{A}]$ for any class K of partial algebras of type τ. The concept of a strong M-hyperidentity of a partial algebra $\underline{A} = (A; (f_i^{\underline{A}})_{i\in I})$ of type τ is defined by

7.1 Definition. Let $\underline{A} = (A; (f_i^{\underline{A}})_{i\in I})$ be a partial algebra of type τ and let $M \subseteq \underline{Hyp_R^c(\tau)}$ be a submonoid of the monoid of all R-hypersubstitutions of type τ. Then a strong identity $t \approx t'$ of $\underline{A}$ is called a *strong M-hyperidentity* of $\underline{A}$ if for every hypersubstitution $\sigma \in M$ the equations $\hat{\sigma}[t] \approx \hat{\sigma}[t']$ are also strong identities of $\underline{A}$. If $t \approx t'$ is a strong M-hyperidentity of $\underline{A}$ we will write $\underline{A} \models_{sM} t \approx t'$. In the case $\underline{M} = \underline{Hyp_R^c(\tau)}$ we will speak of a *strong hyperidentity* of $\underline{A}$.

The following Lemma is helpful to calculate strong hyperidentities.

7.2 Lemma. *Let $V \subseteq PAlg(\tau)$ be a strong variety of partial algebras and let $\sigma_1, \sigma_2 \in Hyp_R^c(\tau)$ be R-hypersubstitutions of type τ with*

$$\sigma_1(f_i) \approx \sigma_2(f_i) \in Id_s V \quad \text{for all} \quad i \in I.$$

Then for every term $t \in W_\tau^c(X)$ we get

$$\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t] \in Id_s V.$$

Proof. We are proving this identity using complexity of the definition of the term t.

If $t = x_i$ is a variable ($x_i \in X_n$) then

$$\hat{\sigma}_1[x_i] = x_i \approx x_i = \hat{\sigma}_2[x_i].$$

Assume now that $t = \varepsilon_j^n(t_1, \ldots, t_k)$ and that $\hat{\sigma}_1[t_i] \approx \hat{\sigma}_2[t_i]$ for every $i \in \{1, \ldots, k\}$. Then

$$\begin{aligned}\hat{\sigma}_1[t] &= \hat{\sigma}_1[\varepsilon_j^k(t_1, \ldots, t_k)] = \overline{S}_n^k(\varepsilon_j^k(x_1, \ldots, x_k), \hat{\sigma}_1[t_1], \ldots, \hat{\sigma}_1[t_k]) \\ &\approx \overline{S}_n^k(\varepsilon_j^k(x_1, \ldots, x_k), \hat{\sigma}_2[t_1], \ldots, \hat{\sigma}_2[t_k]) = \hat{\sigma}_2[\varepsilon_j^k(t_1, \ldots, t_k)] = \hat{\sigma}_2[t].\end{aligned}$$

If finally $t = f_i(t_1, \ldots, t_{n_i})$ and assume again that $\hat{\sigma}_1[t_j] \approx \hat{\sigma}_2[t_j]$ for every $j \in \{1, \ldots, n_i\}$ then

$$\begin{aligned}\hat{\sigma}_1[t] = \hat{\sigma}_1[f_i(t_1, \ldots, t_{n_i})] &= \overline{S}_n^{n_i}(\sigma_1(f_i), \hat{\sigma}_1[t_1], \ldots, \hat{\sigma}_1[t_{n_i}]) \\ &\approx \overline{S}_n^{n_i}(\sigma_1(f_i), \hat{\sigma}_2[t_1], \ldots, \hat{\sigma}_2[t_{n_i}]).\end{aligned}$$

and

$$\begin{aligned}\hat{\sigma}_2[t] = \hat{\sigma}_2[f_i(t_1, \ldots, t_{n_i})] &= \overline{S}_n^{n_i}(\sigma_2(f_i), \hat{\sigma}_2[t_1], \ldots, \hat{\sigma}_2[t_{n_i}]) \\ &\approx \overline{S}_n^{n_i}(\sigma_1(f_i), \hat{\sigma}_2[t_1], \ldots, \hat{\sigma}_2[t_{n_i}])\end{aligned}$$

since $\sigma_1(f_i) \approx \sigma_2(f_i)$ and $Var(\sigma_1(f_i)) = Var(\sigma_2(f_i))$.

This gives us $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$.

Consider the following example:

Let $\underline{2}_f = (\{0,1\}; f)$ be a two-element partial algebra of type $\tau = (2)$ where f is defined by the table

x	y	$f(x,y)$
0	0	not defined
0	1	0
1	0	0
1	1	1

Then it is easy to check that the partial operation f is commutative and associative, i.e. the commutative and the associative law are strong identities of $\underline{2}_f$.

Further, $f(x,x) \approx f(x, f(x,x))$ is a strong identity of $\underline{2}_f$.

We will check whether the associative law $F(x, F(y,z)) \approx F(F(x,y),z)$ is a strong hyperidentity of $\underline{2}_f$. For F we have to substitute any binary term containing both variables.

By Lemma 7.2 we need only to substitute any binary term from the quotient set $W_\tau^c(X)/Id_s\underline{2}_f$. Because of our strong identities $\Sigma = \{f(x,y) \approx f(y,x), f(f(x,y),z) \approx f(x,f(y,z)), f(x,x) \approx f(x,f(x,x))\} \subseteq Id_s\underline{2}_f$ we can substitute any represent from $W_\tau^c(X)/\Sigma \supseteq W_\tau^c(X)/Id_s\underline{2}_f$, i.e. $t_1(x,y) = f(x,y)$, $t_2(x,y) = f(f(x,x),y)$, $t_3(x,y) = f(x,f(y,y))$, $t_4(x,y) = f(f(x,x),f(y,y))$. Then we get the following equations:

$$\begin{aligned}f(f(x,y),z) &\approx f(x,f(y,z)) \\ f(f(f(f(x,x),y),f(f(x,x),y)),z) &\approx f(f(x,x),f(f(y,y),z)) \\ f(f(x,f(y,y)),(z,z)) &\approx f(x,f(f(y,f(z,z)),f(y,f(z,z)))).\end{aligned}$$

$$\begin{aligned}&f(f(x,x),f(f(f(y,y),f(z,z)),f(f(y,y),f(z,z)))) \\ &\qquad \approx f(f(f(f(x,x),f(y,y)),f(f(x,x),f(y,y))),f(z,z))\end{aligned}$$

Using the strong identities from Σ we see that each of these equations is a strong identity, so the associative law is a strong hyperidentity in $\underline{A}$.

Because of $\underline{A} \not\models f(x,f(y,y)) \approx f(y,f(x,x))$ the commutative law is no strong hyperidentity in $\underline{A}$.

For a class K of partial algebras of type τ and for a set $\Sigma \subseteq W_\tau^c(X) \times W_\tau^c(X)$ we define

$H_M Id_s K$ - set of all strong M-hyperidentities satisfied by every partial algebra from K

$H_M Mod_s \Sigma$ - class of all partial algebras of type τ for which every equation from Σ is satisfied as a strong M-hyperidentity.

$H_M Mod_s \Sigma$ is called strong M-hyperequational class defined by Σ.

The operators χ_M^A, χ_M^E have the following property:

7.3 Theorem. *For every submonoid $M \subseteq Hyp_R^c(\tau)$ the operators χ_M^A, χ_M^E form a conjugate pair of additive closure operators with respect to the relation* $\models_s$.

Proof. By its definition as unions the operators χ_M^A, χ_M^E are additive. We are going to prove that they are closure operators. If $t \approx t' \in \Sigma$ then with the hypersubstitution $\sigma_{id} \in M$ we get

$$\hat{\sigma}_{id}[t] = t \approx t' = \hat{\sigma}_{id}[t'] \in \chi_M^E[\Sigma],$$

so $\Sigma \subseteq \chi_M^E[\Sigma]$, i.e. χ_M^E is extensive.

From the additivity we obtain the monotony of χ_M^E, i.e.

$$\Sigma_1 \subseteq \Sigma_2 \Rightarrow \chi_M^E[\Sigma_1] \subseteq \chi_M^E[\Sigma_2].$$

The extensivity of χ_M^E implies $\chi_M^E[\Sigma] \subseteq \chi_M^E[\chi_M^E[\Sigma]]$.

Assume now that $u \approx v \in \chi_M^E[\chi_M^E[\Sigma]]$. Then there are hypersubstitutions $\sigma_1, \sigma_2 \in M$ and an equation $t \approx t' \in \Sigma$ with $u = \hat{\sigma}_1[\hat{\sigma}_2[t]]$ and $v = \hat{\sigma}_1[\hat{\sigma}_2[t']]$, i.e. $u = (\hat{\sigma}_1 \circ \hat{\sigma}_2)[t]$ and $v = (\hat{\sigma}_1 \circ \hat{\sigma}_2)[t']$. By Lemma 6.2 we have $u = (\hat{\sigma}_1 \circ \sigma_2)^{\hat{}}[t]$ and $v = (\hat{\sigma}_1 \circ \sigma_2)^{\hat{}}[t']$ and by definition of the operation $\circ_h$, $u = (\sigma_2 \circ_h \sigma_1)^{\hat{}}[t]$ and $v = (\sigma_2 \circ_h \sigma_1)^{\hat{}}[t']$. Since $\underline{M}$ is a submonoid of $Hyp_R^c(\tau)$ we now have a hypersubstitution $\sigma \in M$, namely $\sigma = \sigma_2 \circ_h \sigma_1$ such that $v = (\sigma_2 \circ_h \sigma_1)^{\hat{}}[t']$ and $u = (\sigma_2 \circ_h \sigma_1)^{\hat{}}[t]$ and thus $u \approx v \in \chi_M^E[\Sigma]$. This shows the idempotency of χ_M^E.

For any two classes K_1, K_2 of algebras of type τ we have $K_1 \subseteq \chi_M^A[K_1]$ and $K_1 \subseteq K_2 \Rightarrow \chi_M^A[K_1] \subseteq \chi_M^A[K_2]$. Then $\chi_M^A[K_1] \subseteq \chi_M^A[\chi_M^A[K_1]]$ is also clear and we have only to show that $\chi_M^A[\chi_M^A[K]] \subseteq \chi_M^A[K]$. Assume that $\underline{B} \in \chi_M^A[\chi_M^A[K]]$. Then there is an algebra $\underline{A} \in K$ and hypersubstitutions $\sigma_1, \sigma_2 \in M$ with $\underline{B} = \sigma_1[\sigma_2[\underline{A}]] = (A; (\sigma_1(f_i)^{\sigma_2[\underline{A}]})_{i \in I})$.

Using Lemma 6.4 we can substitute $(\sigma_1(f_i))^{\sigma_2[\underline{A}]}$ by $(\hat{\sigma}_2[\sigma_1(f_i)])^{\underline{A}}$ and with the definition of $\circ_h$ we have $\hat{\sigma}_2[\sigma_1(f_i)])^{\underline{A}} = ((\sigma_1 \circ_h \sigma_2)(f_i))^{\underline{A}}$.

Since $\sigma_1 \circ_h \sigma_2 \in M$ the algebra $\underline{B} = (A; ((\sigma_1 \circ_h \sigma_2)(f_i)^{\underline{A}})_{i \in I})$ belongs to $\chi_M^A[K]$. So, we have the idempotency.

Putting in Definition 1.1 $\gamma_1 := \chi_M^A$, $\gamma_2 := \chi_M^E$, $A := PAlg(\tau)$, $B := W_\tau^c(X) \times W_\tau^c(X)$ and $R := \models_s$ for the conjugacy we have to prove:

$$\forall \underline{A} \in PAlg(\tau)\ \forall t \approx t' \in W_\tau^c(X) \times W_\tau^c(X)\ (\chi_M^A[\underline{A}] \models_s t \approx t' \Leftrightarrow \underline{A} \models_s \chi_M^E[t \approx t'])$$

Let σ be a hypersubstitution from M and assume that $\sigma[\underline{A}] \models_s t \approx t'$. By Definition 3.1 this means, $t^{\sigma[\underline{A}]} = t'^{\sigma[\underline{A}]}$ for the term operations induced by the terms t and t' on the algebra $\sigma[\underline{A}]$. Since σ is a regular hypersubstitution we can apply Lemma 6.4 and obtain the equation $\hat{\sigma}[t]^{\underline{A}} = \hat{\sigma}[t']^{\underline{A}}$ and therefore $\underline{A} \models_s \hat{\sigma}[t] \approx \hat{\sigma}[t']$. In the same manner we obtain the other direction. ∎

8 M-solid varieties of partial algebras

Since (χ_M^A, χ_M^E) for every submonoid $\underline{M} \subseteq \underline{Hyp_R^c(\tau)}$ is a conjugate pair of additive closure operators we may apply Theorem 1.2, Theorem 1.3, and Proposition 1.4. At first we define

8.1 Definition. Let $V \subseteq PAlg(\tau)$ be a strong variety of partial algebras (i.e. there is a set $\Sigma \subseteq W_\tau^c(X) \times W_\tau^c(X)$ with $V = Mod_s\Sigma$).

Then V is said to be *M-solid* if $\chi_M^A[V] = V$.

Clearly, we have to set $T := V \subseteq PAlg(\tau)$, $S := \Sigma \subseteq W_\tau^c(X) \times W_\tau^c(X)$, $\mu(T) := Id_s V$, $\iota(S) := Mod_s\Sigma$, $\mu_\gamma(T) := H_M\, Id_s V$, $\iota_\gamma(S) = H_M\, Mod_s \Sigma$, $\gamma_1 := \chi_M^A$, $\gamma_2 = \chi_M^E$. All presumptions of Theorem 1.2 are satisfied and we obtain the following characterization of M-solid varieties of partial algebras:

8.2 Theorem. *Let $V \subseteq PAlg(\tau)$ be a strong variety of partial algebras and let $\Sigma \subseteq W_\tau^c(X) \times W_\tau^c(X)$ be a strong equational theory. Then the following propositions* (i) - (iv) and (j) - (jv) are equivalent:

(i) *V is a strong M-hyperequational class, i.e.* $V = H_M\, Mod_s H_M Id_s V$,

(ii) *V is M-solid, i.e.* $\chi_M^A[V] = V$.

(iii) $Id_s V = H_M Id_s V$, *i.e. every strong identity of V is a strong M-hyperidentity*

(iv) $\chi_M^E[Id_s V] = Id_s V$

and

(j) $\Sigma = H_M Id_s H_M\, Mod_s \Sigma$,

(jj) $\chi_M^E[\Sigma] = \Sigma$,

(jjj) $Mod_s \Sigma = H_M\, Mod_s \Sigma$,

(jv) $\chi_M^A[Mod_s \Sigma] = Mod_s \Sigma$.

Note that the equivalence of (i) and (ii) is a Birkhoff-type-characterization of M-solid varieties of partial algebras. The equivalence of (j) and (jj) characterizes strong M-hyperequational theories, that is, strong equational theories closed under the operator χ_M^E.

From Theorem 1.3 we obtain.

8.3 Theorem. *For all $K \subseteq PAlg(\tau)$ and for all $\Sigma \subseteq W_\tau^c(X) \times W_\tau^c(X)$ there holds:*

(i) $\chi_M^A[K] \subseteq Mod_s Id_s K \Rightarrow Mod_s Id_s K = H_M\, Mod_s H_M Id_s K$ and

(ii) $\chi_M^E[\Sigma] \subseteq Id_s\, Mod_s \Sigma \Rightarrow Id_s\, Mod_s \Sigma = H_M Id_s H_M\, Mod_s \Sigma$

Since (Mod_s, Id_s) is a Galois-connection between $PAlg(\tau)$ and $W_\tau^c(X) \times W_\tau^c(X)$ the class of all strong varieties of partial algebras of type τ forms a lattice $\mathcal{L}_{par}(\tau)$. Applyling Proposition 1.4 we get

8.4 Proposition. *The class $S_{M\,par}(\tau)$ of all M-solid varieties of partial algebras of type τ forms a complete sublattice of the lattice $\mathcal{L}_{par}(\tau)$. For any two submonoids $M_1 \subseteq M_2 \subseteq Hyp_R^c(\tau)$ we have $S_{M_1 par}(\tau) \supseteq S_{M_2 par}(\tau)$.*

Proof. The first part is a consequence of Proposition 1.4. If $V \in S_{M_2 par}(\tau)$ then $\chi_{M_2}^A[V] = V$, i.e. for every $\sigma \in M_2$, $\sigma[V] = V$. But then we have also $\sigma[V] \subseteq V$ for all $\sigma \in M_1$ and thus $\chi_{M_1}^A[V] \subseteq V$ and $V \in S_{M_1 par}(\tau)$ ∎

9 Hypersubstitutions of type τ

Because of our new term definition we have to define hypersubstitutions in a slightly different way as in the total case.

9.1 Definition. Let $\{f_i \mid i \in I\}$ be a set of operation symbols of type τ and let $W_\tau^c(X)$ be the set of all terms of this type.

A mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau^c(X)$ which preserves the arity is called *hypersubstitution of type* τ.

Any hypersubstitution of type τ can be extended to terms in the following inductive way. Here $\hat{\sigma} : W_\tau^c(X) \to W_\tau^c(X)$ is the extension of the hypersubstitution σ.

(i) $\hat{\sigma}[x_i] = x_i$ for every $x_i \in X_n$,

(ii) $\hat{\sigma}[\epsilon_j^k(s_1, \ldots, s_k)] = \overline{S}_n^k(\epsilon_j^k(x_1, \ldots, x_k), \hat{\sigma}[s_1], \ldots, \hat{\sigma}[s_k])$, $s_1, \ldots, s_k \in W_\tau^c(X_n)$,

(iii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] = \overline{S}_n^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, $t_1, \ldots, t_{n_i} \in W_\tau^c(X_n)$.

Since the extension $\hat{\sigma}$ of a hypersubstitution σ preserves arities, very extensio $\hat{\sigma}$ defines a family of mappings $\hat{\sigma} := (\eta^{(n)} : W^c_\tau(X_n) \to W^c_\tau(X_n))_{n\in\mathbb{N}}$. W prove

5.2 Theorem. *The extension $\hat{\sigma}$ of a hypersubstitution σ of type τ defines a endomorphism $(\eta^{(n)})_{n\in\mathbb{N}}$ of the term clone* clone τ.

Proof. At first we prove that $(\eta^{(n)})_{n\in\mathbb{N}}$ preserves the constant operatio e^k_j. Indeed, $\eta^{(k)}(e^k_j) = \eta^{(k)}(x_j) = \hat{\sigma}[x_j] = x_j = e^k_j$.

Now we are going to prove the compatibility with the operations $\overline{S}^m_n$ b induction on the complexity of the term definition of $s \in W^c_\tau(X_m)$ i $\overline{S}^m_n(s, t_1, \ldots, t_m)$, $t_1, \ldots, t_m \in W^c_\tau(X_n)$. If s an m-ary variable x_j, $1 \le j \le m$ then we have

$$\begin{aligned}\eta^{(n)}(\overline{S}^m_n(x_j, t_1, \ldots, t_m)) &= \hat{\sigma}[t_j] = \overline{S}^m_n(\hat{\sigma}[x_j], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_m]) \\ &= \overline{S}^m_n(\eta^{(m)}(x_j), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)).\end{aligned}$$

If $s = e^k_j(s_1, \ldots, s_k)$ and assume that for $s_1, \ldots, s_k \in W^c_\tau(X_m)$ the permutabil ity condition is satisfied then

$$\begin{aligned}&\eta^{(n)}(\overline{S}^m_n(e^k_j(s_1, \ldots, s_k), t_1, \ldots, t_m)) = \hat{\sigma}[\overline{S}^m_n(e^k_j(s_1, \ldots, s_k), t_1, \ldots, t_m)] \\ &= \hat{\sigma}[e^k_j(\overline{S}^m_n(s_1, t_1, \ldots, t_m), \ldots, \overline{S}^m_n(s_k, t_1, \ldots, t_m))] \\ &= \overline{S}^k_n(e^k_j(x_1, \ldots, x_k), \hat{\sigma}[\overline{S}^m_n(s_1, t_1, \ldots, t_m)], \ldots, \hat{\sigma}[\overline{S}^m_n(s_k, t_1, \ldots, t_m)]) \\ &= \overline{S}^k_n(e^k_j(x_1, \ldots, x_k), \overline{S}^m_n(\eta^{(m)}(s_1), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)), \ldots, \\ &\qquad \overline{S}^m_n(\eta^{(m)}(s_k), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m))) \\ &= e^k_j(\overline{S}^k_n(x_1, \overline{S}^m_n(\eta^{(m)}(s_1), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)), \ldots, \\ &\qquad \overline{S}^m_n(\eta^{(m)}(s_k), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m))), \ldots, \\ &\qquad \overline{S}^k_n(x_k, \overline{S}^m_n(\eta^{(m)}(s_1), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)), \ldots, \\ &\qquad \overline{S}^m_n(\eta^{(m)}(s_k), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)))) \\ &= e^k_j(\overline{S}^m_n(\eta^{(m)}(s_1), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)), \ldots, \overline{S}^m_n(\eta^{(m)}(s_k), \eta^{(n)}(t_1), \ldots, \eta^{(n)} \\ &= \overline{S}^m_n(e^k_j(\eta^{(m)}(s_1), \ldots, \eta^{(m)}(s_k)), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)).\end{aligned}$$

Now assume that $s = f_i(s_1, \ldots, s_{n_i})$ and further we assume that for $s_1, \ldots, s_{n_i} \in W^c_\tau(X_m)$ the permutability condition is satisfied. Then

$$\begin{aligned}&\eta^{(n)}(\overline{S}^m_n(s, t_1, \ldots, t_m)) = \eta^{(n)}(\overline{S}^m_n(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_m) \\ &= \hat{\sigma}[f_i(\overline{S}^m_n(s_1, t_1, \ldots, t_m), \ldots, \overline{S}^m_n(s_{n_i}, t_1, \ldots, t_m))] \\ &= \overline{S}^{n_i}_n(\sigma(f_i), \hat{\sigma}[\overline{S}^m_n(s_1, t_1, \ldots, t_m)], \ldots, \hat{\sigma}[\overline{S}^m_n(s_{n_i}, t_1, \ldots, t_m)]) \\ &= \overline{S}^{n_i}_n(\eta^{(n_i)}(f_i(x_1, \ldots, x_{n_i})), \overline{S}^m_n(\eta^{(m)}(s_1), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)), \ldots \\ &\qquad \overline{S}^m_n(\eta^{(m)}(s_{n_i}), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m))) \\ &= \overline{S}^m_n(\eta^{(m)}(f_i(s_1, \ldots, s_{n_i})), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)) \\ &= \overline{S}^m_n(\eta^{(m)}(s), \eta^{(n)}(t_1), \ldots, \eta^{(n)}(t_m)).\end{aligned}$$

■

Hypersubstitutions can be applied to strong identities of partial algebras A and we are asking whether the resulting equations $\hat{\sigma}[t] \approx \hat{\sigma}[t']$ are again strong identities of A. Hypersubstitutions can also be applied to partial algebras of type τ and we obtain partial algebras $\sigma[A] := (A; (\sigma(f_i)^A)_{i\in I})$. Based on this definitions one defines two closure operators χ^E and χ^A. As mentioned in the introduction in the total case this is a conjugate pair of additive closure operators what has farreaching consequences. In the partial case this is not longer true since in general the following condition is not satisfied

$$A \models_s \hat{\sigma}[t] \approx \hat{\sigma}[t'] \Leftrightarrow \sigma[A] \models_s t \approx t' \qquad (C)$$

as the following example shows:
Let $A = (A; f^A, n^A)$, $|A| > 1$, be a partial algebra of type $\tau = (2, 1)$ where f^A is an arbitrary binary operation and where n^A is the unary operation with empty domain.

Clearly,

$$A \models_s f(x_1, n(x_2)) \approx f(x_2, n(x_1)) \qquad (1)$$

Consider the hypersubstitution σ defined by $\sigma(f) = x_1$, $\sigma(n) = n(x_1)$. Then applying σ to the Algebra A we have $\sigma[A] = (A; \sigma(f)^A, \sigma(n)^A) =$

$(A; e_1^{2,A}, n^A)$ and $\sigma[\underline{A}] \models_{\tau} (1)$ since $f(x_1, n(x_2))^A = e_1^{2,A}(e_1^{2,A}, n^A)$ is nowher defined and $f(x_2, n(x_1))^A = e_1^{2,A}(e_2^{2,A}, n^A)$ is also nowhere defined.

Further we have $\hat{\sigma}[f(x_1, n(x_2))] = \overline{S}_2^2(\sigma(f), x_1, n(x_2)) = x_1$ an $\hat{\sigma}[f(x_2, n(x_1))] = \overline{S}_2^2(\sigma(f), x_2, n(x_1)) = x_2$. This shows $\underline{A} \not\models_{\tau} \hat{\sigma}[f(x_1, n(x_2))]$ $\hat{\sigma}[f(x_2, n(x_1))]$ since $|A| > 1$.

It turns out that the proper reason for the non-satisfaction of condition (C is that the set of variables occuring in $\hat{\sigma}[f(x_1, x_2)]$ is different from $\{x_1, x_2\}$ So, we have to consider so-called regular hypersubstitutions.

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FINITE COVERS WITH LINKING EQUIVALENCE RELATIONS

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0 Preliminaries

All our definitions are the same as those in [4, 5]

Definition 0.1 *A structure M is a cover of a structure W if there is a surjection $\pi : M \to W$ such that the automorphism group of the structure (W, M, π) induces $Aut(M)$ on M and $Aut(W)$ on W. In particular, the corresponding homomorphism $Aut(M) \to Aut(W)$ is surjective. A set of the form $\pi^{-1}(a), a \in W$, (denoted by $M(a)$) is called a fibre of π. If all fibres are finite then M is a finite cover of W.*

The results of the paper concern finite covers. We always assume that if M covers W then W is countable, ω-categorical and transitive. The latter implies that M can be presented as $W \times F$ for some set F and then π becomes the natural projection $W \times F \to W$. In this case we identify M with the pair of structures $(W, W \times F)$. Also, if W is transitive then the group induced on $M(a), a \in W$, by $Aut(M)$ does not depend on a. We consider it as a permutation group on F and call it the *fibre group* of M. We denote by $Aut(A/B), A, B \subset M$, the set of all permutations on A which extend to automorphisms of M fixing B pointwise. Then the fibre group is $Aut(M(a)/\emptyset)$.

The *kernel* $Ker(M)$ of a cover $M \to W$ is the kernel of the corresponding map $Aut(M) \to Aut(W)$. It is clear that $Ker(M) = Aut(M/W)$. The cover is *linked* if

$$Aut(M(a)/\bigcup\{M(a') : a' \neq a, a' \in W\}) = 1$$

for all $a \in W$. The cover is *superlinked* if $Ker(M)$ is finite.

It is well-known that if N is a countably categorical structure then the definable relations of N are recovered by the action of $Aut(N)$ on N. We also consider $Aut(N)$ as a closed subgroup of $Sym(N)$ under the topology given by cosets of poinwise stabilisers of finite sets (in this topology any closed subgroup is the automorphism group of some structure on N). If $\pi : M \to W$ is a finite cover then the induced homomorphism $\tau : Aut(M) \to Aut(W)$ maps closed (open) subgroups of $Aut(M)$ to closed (open) subgroups of $Aut(W)$ (see [4], Lemma 1.4.2). Also $Ker(M)$ is closed and compact.

A cover $M = (W, W \times F)$ *splits* if it has a covering expansion with trivial kernel. M is *strongly split* if it has a covering expansion with trivial fibre group. It is convenient to reformulate this definition as follows. A *transversal* of $M = (W, W \times F)$ is a subset of $W \times F$ having one-element intersection with each fibre.

> The cover M is strongly split if and only if there is a partition $\{P_f : f \in F\}$ of M by transversals such that the expansion $(M, P_f)_{f \in F}$ still covers W.

Indeed, if M' is an expansion of M with trivial fibre group then $\{P_f : f \in F\}$ is just the set of all 1-orbits under $Aut(M')$.

From the group-theoretical viewpoint split and strongly split covers do not differ at all. In both cases $Aut(M)$ is just a split extension of $Ker(M)$. Some difference arises if we regard covers as permutation groups. The following lemma is given in [5] but the idea, in fact, appears in [2, 3] (concerning untwisted covers). We give the proof here because it will be applied later.

Lemma 0.2 *([5]) A transitive structure W has a split superlinked cover which is not strongly split if and only if $Aut(W/w)$ has a proper closed subgroup of finite index.*

Proof. To show the necessity, notice that if a transitive structure W has a split but not strongly split finite cover then there is a finite cover M' of W which is not strongly split and has trivial kernel. This induces an action of $Aut(W)$ on M'. Let $w \in W$ and $\bar{a}$ be an enumeration of $M'(w)$. Then $Aut(W/\bar{a})$ is a closed subgroup of $Aut(W/w)$ of finite index. Since the fibre group of M' is not trivial, $Aut(W/\bar{a}) \neq Aut(W/w)$.

To show the sufficiency assume that $H < Aut(W/w)$ is closed and of finite index. Take the coset space C^* of $Aut(W)$ on H. We consider C^* under the structure induced by the action of $Aut(W)$. It is easy to show that the map $\rho : C^* \to W$ given by $\rho(gH) = g(w)$ is a finite cover with trivial kernel. The fibre group is non-trivial. $\square$

To some extent the results of the paper concern the situation when the structure W has a proper transitive expansion W_0 such that $Aut(W_0)$ is of finite index in $Aut(W)$. Clearly this situation guarantees that $Aut(W/w)$ has a proper closed subgroup of finite index. Indeed, by the transitivity of W_0, $Aut(W/w) \not\subseteq Aut(W_0)$. Then the group $H = Aut(W_0) \cap Aut(W/w)$ is of finite index in $Aut(W/w)$.

In general, let W be a transitive ω-categorical structure, $w \in W, G = Aut(W)$ and G_0 be a closed normal subgroup of G of finite index such that $Aut(W/w) \not\subseteq G_0$. Let C^* be the coset space of G on $H = G_0 \cap Aut(W/w)$ and $\rho : C^* \to W$ be the cover defined in the proof of Lemma 0.2. It is clear that the fibre group of this cover is not trivial. Let E be the equvalence relation on C^* consisting of the left cosets of G_0. It is easy to see that E is a definable finite equivalence relation. Observe that the intersection of any fibre of ρ with any E-class has at most one element. We have arrived at the following definition.

Definition 0.3 *Let $\pi : M \to W$ be a finite cover of a transitive countably categorical structure W. Let E be a finite equivalence relation on M definable over $\emptyset$. We say that E is linking if the intersection of any E-class with any fibre has most one element.*

It is easily seen that a cover having a linking equivalence relation is superlinked. The equivalence relation from Theorem 5.1.4 of [4] is linking. In the rest of the paper we show that finite covers with a linking equivalence relation are completely determined by finite structures which naturally arise in this situation.

Finite structures

Let D be a finite 0-definable equivalence relation (on C) of a finite cover $C \to W$. The group of automorphisms of C fixing the D-classes is normal and of finite index in $G = Aut(C)$. Its image in $H = Aut(W)$ under the natural map $\tau : Aut(C) \to Aut(W)$ also is closed, normal and of finite index in H. Thus it is defined as a group of automormisms fixing the classes of some definable finite equivalence relation $\hat{D}$ on some W^l. It is easily seen that the group (denoted by $H/\dot{D}$) of the automorphisms of the induced (finite) structure on $W^l/\dot{D}$ is a homomorphic image of the corresponding group G/D on C/D. Let $h_{C,D} : G/D \to H/\dot{D}$ be the corresponding surjection (naturally defined by $Aut(C)$).

Let E be a linking equivalence relation of π. We may assume that E is the finest definable finite equivalence relation on C. Let $\dot{E}$ be the equivalence relation defined as above on the corresponding W^k.

Proposition 1.1 *Any automorphism of W fixing the $\dot{E}$-classes on W^k has a unique extension to an automorphism of C fixing the E-classes.*

The kernel of C is naturally isomorphic to the kernel of $h_{C,E}$.

Proof. Note that for $\alpha \in Ker(C)$ the definition of a linking equivalence relation implies that $\alpha = id$ if α fixes the E-classes. This guarantees that any automorphism of W fixing the $\dot{E}$-classes on W^k has a unique extension to an automorphism of C fixing the E-classes.

It is easily seen that if $\alpha \in Ker(C)$ then the corresponding $\alpha_E \in G/E$ is in $Ker(h_{C,E})$. As we noticed in the previous paragraph the homomorphism $\alpha \to \alpha_E$ is injective for the elements of $Ker(C)$. It remains to show that this homomorphism is surjective on $Ker(C)$.

If $\gamma_E \in Ker(h_{C,E})$ is defined by $\gamma_1 \in Aut(C)$ then $\tau(\gamma_1)$ equals $\tau(\gamma_2)$ for some $\gamma_2 \in Aut(C)$ with trivial action on C/E. Then $\gamma_1 \cdot \gamma_2^{-1} \in Ker(C)$ and $\gamma_E = (\gamma_1 \cdot \gamma_2^{-1})_E$. $\square$

In a sense Proposition 1.1 shows that the homomorphism $h_{C,E}$ determines C. We now characterise splitting in the introduced terms. The case of strongly split covers is completely transparent and will be described in Proposition 1.3.

Proposition 1.2 *The cover C splits if and only if there is a definable fin equivalence relation $\hat{D}$ on some W^l such that the automorphisms of W fix the $\hat{D}$-classes fix the $\hat{E}$-classes and there is an embedding $j : H/\hat{D} \to G$ such that for any $\beta_{\hat{D}} \in H/\hat{D}$ there is $\gamma \in G$ that induces $\beta_{\hat{D}}$ and $j(\beta_{\hat{D}})$ $\gamma_E \in G/E$).*

Proof. Let C split and M be the corresponding trivial expansion of Then E is the finest definable in M finite equivalence relation because a finer equivalence relation induces a finer equivalence relation on W than th induced by E (in fact the set $\pi(E) \subseteq W^2$). Let $G_1 = Aut(M)$. As the case of C we find a definable finite equivalence relation $\hat{D}$ on some and a surjection $h_{M,E} : G_1/E \to H/\hat{D}$. Repeating the proof of Propositi 1.1 we have that $Ker(h_{M,E})$ is trivial ($Ker(M)$ is trivial). This defines t embedding j. It is clear that any $\gamma \in G_1$ induces $j(\gamma_{\hat{D}}) \in G/E$ for t corresponding $\gamma_{\hat{D}} \in H/\hat{D}$.

By Proposition 1.1 any automorphism of W fixing the $\hat{D}$-classes fixes t $\hat{E}$-classes (apply that $Aut(M) \leq Aut(C)$).

For the converse let G_2 be the group of all automorphism of C fixing t E-classes and the classes of $\hat{D}$ on W^l. By Proposition 1.1 any of them uniquely determined by its restriction on W. It is clear that G_2 is close Now for every $\gamma_{\hat{D}} \in H/\hat{D}$ choose some $\gamma \in Aut(C)$ inducing $j(\gamma_{\hat{D}}) \in G/$ and $\gamma_{\hat{D}}$.

Notice that each automorphism of W has an extension in $\gamma \cdot G_2$ for some these γ's. Indeed let $\beta \in Aut(W)$ induce $\beta_{\hat{D}} \in H/\hat{D}$. Take the $\gamma \in Aut(C$ chosen for $\beta_{\hat{D}}$ as above. Then $\beta^{-1} \cdot \gamma$ preserves the $\hat{D}$-classes on W^l. The it preserves the $\hat{E}$-classes on W^k and extends to some $\alpha \in G_2$. So β extend to $\gamma \cdot \alpha^{-1} \in \gamma \cdot G_2$.

If γ, γ' and γ'' are the chosen automorphisms for $\gamma_{\hat{D}}, \gamma'_{\hat{D}}$ and $\gamma''_{\hat{D}}$ with $\gamma_{\hat{D}}$ $\gamma'_{\hat{D}} = \gamma''_{\hat{D}}$ then $(\gamma \cdot \gamma')^{-1} \cdot \gamma''$ preserves the E-classes and the $\hat{D}$-classes. Thi shows that the union of the chosen cosets is a group G_1 (the case of $^{-1}$ i similar). It is straightforward that G_1 is closed (apply that G_2 is closed). Hence G_1 induces a cover of W. This cover is trivial because the kernel of G_1 is a subgroup of G_2. □

Note that the set $\pi(E) \subset W^2$ is the finest definable equivalence relation on W and the natural map $C/E \to W/\pi(E)$ is a cover of finite structures where the size of a fibre is the same as that for π. Thus if $\hat{E}$ is an equivalence relation on W then $\hat{E}$ is exactly $\pi(E)$, and the corresponding map $C/E \to W/\hat{E}$ is a cover of finite structures.

Proposition 1.3 *The cover C is strongly split if and only if all automorphisms of W preserving the $\pi(E)$-classes preserve the $\hat{E}$-classes and the cover $C/E \to W/\pi(E)$ is strongly split.*

Proof. If C is strongly split then take a covering expansion M with trivial fibre group. Note that each $\beta \in Aut(W)$ preserving the $\pi(E)$-classes extends to an automorphism of M preserving the E-classes (any transversal determined by M is a copy of W). So β preserves the $\hat{E}$-classes. The cover $M/E \to W/\pi(E)$ defines an expansion of $C/E \to W/\pi(E)$ with trivial fibre group.

For the converse take a family $\{P_i : i \leq |\pi^{-1}(w)|\}$ of transversals of C/E determining a covering expansion of $C/E \to W/\pi(E)$ with trivial fibre group. For each P_i the union of the E-classes from P_i is a transversal P_i^* of C. Since the expansion by the transversals P_i is a cover of $W/\pi(E)$, the automorphisms of C preserving all P_i^* induce all automorphisms of $W/\pi(E)$. To show that the expansion of C by the transversals P_i^* is a cover of W it suffices to prove that any automorphism of W preserving the $\pi(E)$-classes extends to an automorphism of C preserving the E-classes. This follows from the fact that any automorphism preserving the $\pi(E)$-classes preserves the $\hat{E}$-classes. □

2 Example

In this section we illustrate Proposition 1.2. So we give an example of a cover with a linking equivalence relation such that the cover is split but not strongly split.

Let $(\mathbf{Q}, <)$ be the ordering of the rationals and $\mathbf{Q}_0$ be a dense and co-dense subset of $\mathbf{Q}$. Consider the equivalence relation E_0 with the classes $\mathbf{Q}_0$ and

$\mathbf{Q}\setminus\mathbf{Q}_0$. Let W be the structure $(\mathbf{Q}, E_0, B)$ where B is a ternary betweenne relation defined by: $B(x,y,z) \leftrightarrow (y < x < z) \vee (z < x < y)$. It is well-know that $Aut(\mathbf{Q}, B) = Aut(\mathbf{Q}, <) \cup \delta \cdot Aut(\mathbf{Q}, <)$, where the automorphism δ defined by $\delta(q) = -q$ (see [1]). We assume that $\mathbf{Q}_0$ is closed under the acti of δ.

Let $M = W \times \{0,1,2,3\}$ and E be the equivalence relation on M consisti of the E_0-classes of the transversals $P_i = \{(w,i) : w \in W\}$ (thus E consis of 8 classes). Let E' be the equivalence relation consisting of the transve sals. Then we consider M under E, E' and the betweenness relations on t transversals P_i, that are induced by B on $\mathbf{Q}$. Finally we add a predicate that is the union of the $\mathbf{Q}_0$-classes of P_0 and P_2 and $(\mathbf{Q}\setminus\mathbf{Q}_0)$-classes of and P_3.

Lemma 2.1 *The structure M is a cover of W under the natural projecti The relation E is a linking equivalence relation. The kernel of M is isom phic to $\mathbf{Z}_2 \times \mathbf{Z}_2$. The cover $\pi : M \to W$ is not strongly split.*

Proof. It is clear that any automorphism of W preserving the E_0-clas extends to M by the corresponding action on the transversals. If $\alpha \in Aut(W$ does not preserves the E_0-classes then we can extend it to M by taking onto P_1, P_1 onto P_0 and P_2 onto P_3.

The kernel of M is generated by the inversions $P_0 \to P_2$ and $P_1 \to P_3$.

Any covering expansion of M with trivial fibre group must have una relations for the transversals (E' forces this). In this case we can not exte an automorphism not preserving the E_0-classes. □

We now define a binary relation $<$ on every P_i which is the order of t rationals on P_0 and P_1 (under the projection). On P_2 and P_3 we define $<$ be the reverse order. The resulting expansion M_0 also is a cover of W. see this extend any $\alpha \in Aut(\mathbf{Q}, E_0, <)$ as in the proof of Lemma 2.1. Th automorphism δ can be extended by taking P_0 onto P_2 and P_1 onto P_3. It easily seen that the kernel of M_0 is trivial. So we have:

Lemma 2.2 *The cover $\pi : M \to W$ is split.* □

To illustrate the material of Section 2 note that the relation $\hat{E}$ define there, is $E_0 = \pi(E)$. On the other hand the corresponding cover M/E -

W/E_0 is not strongly split. The equivalence relation $\hat{D}$ obtained by M_0 is defined on W^2 as follows:

$((w_1,w_2),(w_3,w_4)) \in \hat{D}$ if and only if w_1 and w_3 (w_2 and w_4 respectively) are of the same E_0-class and $w_1 < w_2 \leftrightarrow w_3 < w_4$ (and $w_1 > w_2 \leftrightarrow w_3 > w_4$).

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SOME PROPERTIES OF $\aleph_0$-CATEGORICAL WEAKLY O-MINIMAL THEORIES

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1 Preliminaries

Let L be a countable first-order language. Everywhere in this paper we consider L-structures and assume that L contains a binary relation symbol $<$ that is interpreted as a linear ordering in these structures. For arbitrary subsets A, B of a structure M we write $A < B$ if $a < b$ whenever $a \in A$ and $b \in B$. If $A \subset M$ and $x \in M$ then we write $A < x$ if $A < \{x\}$. For an arbitrary complete type p we denote by $p(M)$ the set of realizations of the type p in M.

Definition 1.1 Let $A \subseteq M$, where M is a linearly ordered structure. The set A is said to be *convex* if for any $a, b \in A$ and $c \in M$ such that $a < c < b$ we have $c \in A$.

Definition 1.2 [1] A linearly ordered structure M is said to be *o-minimal* if any definable (with parameters) subset of M is a finite union of intervals in M.

Definition 1.3 [2] A linearly ordered structure M is said to be *weakly o-minimal* if any definable (with parameters) subset of M is a finite union of convex sets in M.

In this paper we use Baizhanov's technique which he has elaborated for classification of one-types in weakly o-minimal theories [3], [4]. In the following definitions M is a weakly o-minimal structure, $A \subseteq M$, $p \in S_1(A)$ is non-algebraic.

Definition 1.4 [3] A formula $F(x,y)$ is said to be *convex to the right (left)* if

$$M \models \forall y \forall x[(F(x,y) \to (y \le x \land \forall z(y \le z \le x \to F(z,y))))]$$

$$(M \models \forall y \forall x[(F(x,y) \to (x \le y \land \forall z(x \le z \le y \to F(z,y))))])$$

Definition 1.5 [3] An A-definable formula $F(x,y)$ is said to be *p-stable* if there are $\alpha, \gamma_1, \gamma_2 \in p(M)$ such that $\gamma_1 < F(M,\alpha) < \gamma_2$.

Definition 1.6 [3] Let $F_1(x,y)$, $F_2(x,y)$ be p-stable convex to the right (left) formulas. We will say $F_2(x,y)$ is *greater than* $F_1(x,y)$ if there is $\alpha \in p(M)$ such that $F_1(M,\alpha) \subset F_2(M,\alpha)$.

Fact 1.7 *[3] Let M be a weakly o-minimal structure, $A \subseteq M$, $p \in S_1(A)$ be non-algebraic. Then*

1. *For any A-definable formulas $F_1(x,y)$, $F_2(x,y)$ if there is $\alpha \in p(M)$ such that $F_1(M,\alpha) \subset F_2(M,\alpha)$ then for any $\beta \in p(M)$ $F_1(M,\beta) \subset F_2(M,\beta)$.*
2. *The set of all p-stable convex to the right (left) formulas is linearly ordered.*

Definition 1.8 [3] We will say p is *semiquasisolitary to the right (left)* if there is the greatest p-stable convex to the right (left) formula.

Definition 1.9 [3] We will say p is *quasisolitary* if p is semiquasisolitary both to the right and to the left.

Definition 1.10 [3] Let $F(x,y)$ be a p-stable convex to the right (left) formula. We will say $F(x,y)$ is *locally p-decreasing (p-increasing)* if there are $\alpha_1, \alpha_2 \in p(M)$ such that

$$M \models \exists x(F(x,\alpha_2) \land \neg F(x,\alpha_1) \land F(\alpha_1,\alpha_2) \land \alpha_1 < x)$$

$$(M \models \exists x(F(x,\alpha_2) \land \neg F(x,\alpha_1) \land F(\alpha_1,\alpha_2) \land \alpha_1 > x))$$

Lemma 1.11 *[3] Let M be a weakly o-minimal structure, $A \subseteq M$, M be $|A|^+$-saturated, $p \in S_1(A)$ be non-algebraic, and let $F(x,y)$ be a p-stable convex to the right (left) formula. Suppose that $F(x,y)$ is locally p-decreasing (p-increasing). Then there is a p-stable convex to the right (left) formula which is greater than $F(x,y)$ and it is not locally p-decreasing (p-increasing).*

Proof of Lemma 1.11.

Without loss of generality suppose that $F(x,y)$ is convex to the right. As $F(x,y)$ is locally p-decreasing there are $\alpha,\beta,\gamma \in p(M)$ such that

$$\alpha<\beta<\gamma,\ \beta \in F(M,\alpha), \gamma \in F(M,\alpha)\setminus F(M,\beta).$$

Let $F_0(\beta,x)$ be a maximal convex subformula of $F(\beta,x)$ such that $\alpha \in F_0(\beta,M)$. Consider the following formula:

$$F_1(x,y) := \exists z[F_0(y,z)\wedge F(x,z)\wedge x\geq y]$$

It is obvious $F_1(x,y)$ is the required formula. □

Definition 1.12 Let $F(x,y)$ be a p-stable convex to the right (left) formula. We will say $F(x,y)$ is *locally p–constant* if for any $\alpha,\beta \in p(M)$ such that $M\models F(\beta,\alpha)$ the following holds:

$$M\models \forall x[x\geq\beta \rightarrow [F(x,\alpha)\leftrightarrow F(x,\beta)]]$$

$$(M\models \forall x[x\leq\beta \rightarrow [F(x,\alpha)\leftrightarrow F(x,\beta)]])$$

Lemma 1.13 *[3] Let M be a weakly o-minimal structure, $A\subseteq M$, M be $|A|^+$-saturated, $p\in S_1(A)$ be non-algebraic. Then*

1. *If $F(x,y)$ is the greatest p–stable convex to the right (left) formula then $F(x,y)$ is locally p–constant.*
2. *Any semiquasisolitary one-type is quasisolitary.*

The following two lemmas are consequences of the proof of Lemma 1.13.

Lemma 1.14 *Let M be a weakly o-minimal structure, $A\subseteq M$, M be $|A|^+$-saturated, $p\in S_1(A)$ be non-algebraic, $F(x,y)$ be a p–stable convex to the right (left) formula. Then the following formula*

$$F'(x,y) := \exists z[F(z,y)\wedge F(x,z)]$$

also is p–stable convex to the right (left).

Lemma 1.15 *Let M be a weakly o-minimal structure, $A\subseteq M$, $p\in S_1(A)$ be non-algebraic, M be $|A|^+$-saturated. Suppose that $F(x,y)$ is a p–stable convex to the right formula so that $F(x,y)$ is locally p–constant. Then*

1) $G(x,y) := F(y,x)$ is a p-stable convex to the left formula which is also locally p–constant.

2) $E(x,y) := F(x,y)\vee F(y,x)$ is an equivalence relation which partitions $p(M)$ on convex classes.

Definition 1.16 Let $F(x,y)$ be a p-stable convex to the right (left) formula. We will say $F(x,y)$ is *trivial* if the following holds:

$$\text{for any } \alpha\in p(M) \quad M\models\forall x\,[\,F(x,\alpha)\rightarrow x=\alpha\,].$$

Otherwise, such a formula is said to be *non-trivial.*

Definition 1.17 [3] A quasisolitary type p is said to be *solitary* if the greatest p-stable convex to the right (left) formula is trivial.

Definition 1.18 [3] A non-isolated type $p\in S_1(A)$ is said to be *quasirational to the right (left)* if there is an A-definable formula $U(x)$ so that for any $\alpha\in p(M)$, $\beta\in U(M)$ such that $\alpha<\beta$ ($\alpha>\beta$) we have $\beta\in p(M)$. Otherwise, it is said to be *irrational to the right (left).*

Definition 1.19 Let M be a linearly ordered structure, $A,B\subseteq M$, $n\in\omega$.

1. We will say A is *n-indiscernible over B* in M if for any properly ordered n-tuples $\bar a$, $\bar b\in A^n$ $tp(\bar a/B)=tp(\bar b/B)$.
2. We will say A is *indiscernible over B* in M if for any $n\in\omega$ A is n-indiscernible over B in M.

Lemma 1.20 *Let M be a weakly o-minimal structure, $A\subseteq M$, $p\in S_1(A)$ be non-algebraic. Suppose that p is solitary. Then $p(M)$ is 2-indiscernible over A.*

Proof of Lemma 1.20.

First we prove the following:

$$\text{for any } \alpha_1,\alpha_2,\alpha_3\in p(M) \text{ such that } \alpha_1<\alpha_2,\ \alpha_1<\alpha_3$$

$$tp(\alpha_1,\alpha_2/A)=tp(\alpha_1,\alpha_3/A) \quad (*)$$

For a contradiction, suppose that there are $\alpha_1,\alpha_2,\alpha_3\in p(M)$ such that $\alpha_1<\alpha_2$, $\alpha_1<\alpha_3$ and $tp(\alpha_1,\alpha_2/A)\neq tp(\alpha_1,\alpha_3/A)$. Then there is an A-definable

formula $\phi(x,y)$ such that $M \models \phi(\alpha_1,\alpha_2) \wedge \neg\phi(\alpha_1,\alpha_3)$. By weak o-minimality of M $\phi(\alpha_1, M)$ is a finite union of convex sets. Then, let $\phi_0(\alpha_1, M)$ be a maximal convex subset of $\phi(\alpha_1, M)$ such that $\alpha_2 \in \phi_0(\alpha_1, M)$.
If $\alpha_2 < \alpha_3$ then consider the following formula:

$$F(x,\alpha_1) := \alpha_1 \le x \wedge \forall y[\phi_0(\alpha_1,y) \to x \le y]$$

It is clear that $M \models F(\alpha_2,\alpha_1) \wedge \neg F(\alpha_3,\alpha_1)$. It can understand that $F(x,y)$ is a p-stable convex to the right formula. This contradicts to a solitarity of p.
If $\alpha_3 < \alpha_2$ then consider the following formula:

$$F(x,\alpha_1) := \alpha_1 \le x \wedge \forall y[\phi_0(\alpha_1,y) \to x < y]$$

It is clear that $M \models F(\alpha_3,\alpha_1) \wedge \neg F(\alpha_2,\alpha_1)$. It can understand that $F(x,y)$ is a p-stable convex to the right formula. This contradicts to a solitarity of p.
Thus, $(*)$ holds.
It can show by analogy that:

for any $\alpha_1,\alpha_2,\alpha_3 \in p(M)$ such that $\alpha_2 < \alpha_1$, $\alpha_3 < \alpha_1$

$$tp(\alpha_1,\alpha_2/A) = tp(\alpha_1,\alpha_3/A) \quad (**)$$

Otherwise, we have a p-stable convex to the left formula. This contradicts to a solitarity of p.
Now, consider two arbitrary 2-tuples of increasing elements from $p(M)$: $\langle\alpha_1,\alpha_2\rangle, \langle\beta_1,\beta_2\rangle$ $(\alpha_1 < \alpha_2,\ \beta_1 < \beta_2)$ and show that their types over A coincide.
Case 1. $\alpha_1 < \beta_2$.
Then according to properties $(*), (**)$ for any A-definable formula $\phi(x,y)$ we have

$$M \models \phi(\alpha_1,\alpha_2) \Leftrightarrow M \models \phi(\alpha_1,\beta_2) \Leftrightarrow M \models \phi(\beta_1,\beta_2).$$

Case 2. $\alpha_1 \ge \beta_2$.
Then according to properties $(*), (**)$ for any A-definable formula $\phi(x,y)$ we have

$$M \models \phi(\alpha_1,\alpha_2) \Leftrightarrow M \models \phi(\beta_1,\alpha_2) \Leftrightarrow M \models \phi(\beta_1,\beta_2).$$

Thus, $p(M)$ is 2-indiscernible over A. □

Lemma 1.21 *Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$, M be $|A|^+$-saturated, $p \in S_1(A)$ be non-algebraic. Suppose that $E(x,y)$ is an A-definable equivalence relation which partitions $p(M)$ on convex classes. Then E partitions $p(M)$ on infinitely many such classes, so that the induced ordering on classes is either a dense order without endpoints or a discrete order without endpoints.*

Proof of Lemma 1.21.
First show that there is no the leftmost E-class containing in $p(M)$. If p is irrational to the left that by $|A|^+$-saturation for any $\beta \in p(M)$ there is $\gamma \in p(M)$ such that $\gamma < E(M,\beta)$. Let p be quasirational to the left or isolated. Then there is an A-definable formula $U(x)$ such that for any $\alpha \in p(M)$, $\gamma \in U(M)$ so that $\alpha > \gamma$ we have $\gamma \in p(M)$. For a contradiction, suppose that there is $\beta \in p(M)$ such that for any $\alpha \in p(M)$ we have $\alpha \in E(M,\beta)$ or $E(M,\beta) < \alpha$. Then consider the following formula:

$$\Theta(x) := U(x) \wedge \exists y E(x,y) \wedge \forall t[(U(t) \wedge t < x) \to E(x,t)]$$

It is clear that $\Theta(M) = E(M,\beta)$, i.e. $E(M,\beta)$ is A-definable. This contradicts to 1-indiscernibility of $p(M)$ over A. We can also show that there is no the rightmost E-class. Thus, E partitions $p(M)$ on infinitely many classes. Now, consider the following formula:

$$\Phi(x) := \exists y E(x,y) \wedge \exists z[\neg E(z,x) \wedge x < z \wedge \forall t(x < t < z \to E(x,t) \vee E(z,t))]$$

If $\Phi(x) \in p$ then E-classes are discretely ordered. If not, then E-classes are densely ordered. □

2 General properties

Fact 2.1 *Let M be a weakly o-minimal structure. Suppose that $Th(M)$ is $\aleph_0$-categorical. Then $Th(M)$ is weakly o-minimal.*

Fact 2.2 *Let T be an $\aleph_0$-categorical weakly o-minimal theory, $M \models T$. Then the set of all elements from M which have either an immediate successor or an immediate predecessor is finite and it is contained in $dcl(\emptyset)$.*

Lemma 2.3 *Let T be an $\aleph_0$-categorical weakly o-minimal theory, $M \models T$. Then there is a finite set $C = \{c_0, \ldots, c_n\} \subseteq M$ ($M \cup \{-\infty, +\infty\}$, if M does not have a first or last element), consisting of all $\emptyset$-definable elements in M (with the possible exceptions of $-\infty, +\infty$), such that $M \models c_i < c_j$ for all $i < j \leq n$ and for each $j \in \{1, \ldots, n\}$ either $M \models \neg(\exists x)c_{j-1} < x < c_j$ or $I_j = \{ x \in M : M \models c_{j-1} < x < c_j \}$ is a dense linear order without endpoints.*

Proof of Lemma 2.3.
Let $C = \{ c \in M : c$ is $\emptyset$-definable in $M \}$. By $\aleph_0$-categoricity T C must be finite. Let C ($\cup\{ -\infty, +\infty \}$, if M does not have a first or last element) be enumerated as $\{ c_0, \ldots, c_n \}$.
Next, suppose that $I_j = \{ x \in M : M \models c_{j-1} < x < c_j \} \neq \emptyset$. Then by Fact 2.2 I_j must be dense without endpoints, completing the proof of the lemma.

□

Fact 2.4 *Let T be an $\aleph_0$-categorical weakly o-minimal theory, $M \models T$, $A \subseteq M, M$ be $|A|^+$-saturated, $p \in S_1(A)$ be non-algebraic. Suppose that $E(x, y)$ is an A-definable equivalence relation which partitions $p(M)$ on convex classes. Then the induced order on E-classes is a dense order without endpoints.*

Fact 2.5 *Let T be an $\aleph_0$-categorical weakly o-minimal theory, $M \models T$, $A \subseteq M, M$ be $|A|^+$-saturated, $p \in S_1(A)$ be non-algebraic. Suppose that $E_1(x, y)$, $E_2(x, y)$ are A-definable equivalence relations which partition $p(M)$ on convex classes so that there is $\alpha \in p(M)$ such that $E_1(M, \alpha) \subset E_2(M, \alpha)$. Then E_1 partitions each E_2-class on infinitely many E_1-classes.*

Fact 2.6 *Let T be an $\aleph_0$-categorical weakly o-minimal theory, $M \models T$, $A \subseteq M$, A be finite, $p \in S_1(A)$. Then there is only finitely many p-stable convex to the right (left) formulas.*

Corollary 2.7 *Let T be an $\aleph_0$-categorical weakly o-minimal theory, $M \models T$, $A \subseteq M$, A be finite. Then any non-algebraic one-type over A is quasisolitary.*

Proposition 2.8 *Let T be an $\aleph_0$-categorical weakly o-minimal theory, $M \models T$, $A \subseteq M$, A be finite, $p \in S_1(A)$ be non-algebraic. Then any p-stable convex to the right (left) formula is locally p-constant.*

Proof of Proposition 2.8.
As T is $\aleph_0$-categorical then there is only finitely many p-stable convex to the right formulas: $F_1(x, y), F_2(x, y), \ldots, F_n(x, y)$, so that for any $\alpha \in p(M)$ we have

$$F_1(M, \alpha) \subset F_2(M, \alpha) \subset \ldots \subset F_n(M, \alpha)$$

Prove that for any $i \in \{1, \ldots, n\}$ $F_i(x, y)$ is locally p-constant.
Step 1. $F_n(x, y)$ is locally p-constant.
This follows from Lemma 1.13.
Step i. Suppose that for any $j \in \{i+1, \ldots, n\}$ $F_j(x, y)$ is locally p-constant.
Prove that $F_i(x, y)$ also is locally p-constant.
For a contradiction, suppose that $F_i(x, y)$ is not locally p-constant.
Case 1. There are $\alpha, \beta, \gamma \in p(M)$ such that

$$M \models F_i(\beta, \alpha) \land F_i(\gamma, \beta) \land \neg F_i(\gamma, \alpha)$$

Consider the following formula:

$$F'(x, y) := \exists t[F_i(t, y) \land F_i(x, t)]$$

From Lemma 1.14 $F'(x, y)$ is p-stable convex to the right. It is clear that $F'(M, \alpha) \supset F_i(M, \alpha)$. If there is $j \in \{i, \ldots, n-1\}$ such that

$$F_j(M, \alpha) \subset F'(M, \alpha) \subset F_{j+1}(M, \alpha),$$

then it contradicts to the hypothesis that $F_{j+1}(x, y)$ is an immediate successor of $F_j(x, y)$ among all p-stable convex to the right formulas. Consequently, there is $j \in \{i+1, \ldots, n\}$ $F'(M, \alpha) = F_j(M, \alpha)$. Then the following holds:

$$M \models \forall x(F_j(x, \alpha) \to (F_i(x, \alpha) \lor \exists t[F_i(t, \alpha) \land F_i(x, t)])) \quad (1)$$

Consider an arbitrary element $\gamma_1 \in F_j(M, \alpha) \setminus F_i(M, \alpha)$. By (1) there is $\beta \in F_i(M, \alpha)$ such that $\gamma_1 \in F_i(M, \beta)$.
Then, by considering of an automorphism $f \in Aut_A(M)$ such that $f(\gamma_1) = \alpha$

we obtain that $F_i(\gamma_1, M)$ is a union of infinitely many convex sets, contr dicting to weak o-minimality of M.
Case 1 is considered completely.
Case 2. There are $\alpha, \beta, \gamma \in p(M)$ such that

$$M \models F_i(\beta,\alpha) \wedge F_i(\gamma,\alpha) \wedge \neg F_i(\gamma,\beta) \wedge \alpha < \beta < \gamma,$$

i.e. $F_i(x,y)$ is locally p-decreasing.
Consider $F_i(\beta, M)$. By weak o-minimality of M $F_i(\beta, M)$ is a union finitely many convex sets. Let $F_i^0(\beta, x)$ is a convex subformula of $F_i(\beta,$ such that $M \models F_i^0(\beta,\alpha)$
It can understand that $F_i^0(\beta, M) \subseteq p(M)$ and there is $\beta' \in p(M)$ such th $\beta' < F_i^0(\beta, M)$, i.e. $F_i^0(x,y)$ is p-stable.
Consider the following formula:

$$F_i'(x,y) := \exists z[F_i^0(y,z) \wedge F_i(x,z) \wedge y \leq x]$$

By Lemma 1.11 $F_i'(x,y)$ is p-stable convex to the right non-locally decreasing and $F_i(x,y)$ is less than $F_i'(x,y)$. If there is $j \in \{i, \ldots, n-1\}$ suc that

$$F_j(M,\beta) \subset F_i'(M,\beta) \subset F_{j+1}(M,\beta),$$

then it contradicts to the hypothesis that $F_{j+1}(x,y)$ is an immediate successor of $F_j(x,y)$ among all p-stable convex to the right formulas. Consequently there is $j \in \{i+1, \ldots, n\}$ such that $F_i'(M,\beta) = F_j(M,\beta)$. Then the following holds:

$$M \models \forall x(F_j(x,\beta) \rightarrow (F_i(x,\beta) \vee \exists t[F_i^0(\beta,t) \wedge F_i(x,t)])) \quad (2)$$

Consider an arbitrary element $\gamma_1 \in F_j(M,\beta) \setminus F_i(M,\beta)$.
By (2) there is $\gamma_2 \in F_i^0(\beta, M)$ such that $M \models F_i(\gamma_1, \gamma_2)$. So we have that $M \models F_j(\gamma_1, \gamma_2)$.
Then, by considering of an automorphism $f \in Aut_A(M)$ such that $f(\gamma_1) = \gamma_2$ we obtain that $F_i(\gamma_1, M)$ is a union of infinitely many convex sets contradicting to weak o-minimality of M.
Case 2 is considered completely.
Step i is proved.

orollary 2.9 *Let T be an $\aleph_0$-categorical weakly o-minimal theory. Then r any $M \models T$ for any finite $A \subseteq M$ for any non-algebraic type $p \in S_1(A)$ ither $p(M)$ is 2-indiscernible over A or there is finitely many A-definable uivalence relations $E_1(x,y), \ldots, E_m(x,y)$ such that*

1. E_m partitions $p(M)$ on infinitely many E_m-classes, each E_m-class is convex and open, so that the induced order on classes is a dense linear order without endpoints.

2. For every $i \in \{1, \ldots, m-1\}$ E_i partitions each E_{i+1}-class on infinitely many E_i-classes, each E_i-class is convex and open, so that E_i-subclasses of each E_{i+1}-class are densely ordered without endpoints.

3. For any $\alpha \in p(M)$ $E_1(M,\alpha)$ is 2-indiscernible over A.

roof of Corollary 2.9.
If type p is solitary then by Lemma 1.20 $p(M)$ is 2-indiscernible over A. Now, uppose that p is not solitary. Then there is only finitely many non-trivial stable convex to the right formulas.
et $\{F_1(x,y), \ldots, F_m(x,y)\}$ be a list of all non-trivial p-stable convex to the ight formulas, so that for any $\alpha \in p(M)$

$$F_1(M,\alpha) \subset \ldots \subset F_m(M,\alpha).$$

By Proposition 2.9 all of them are locally p-constant. Then by Lemma 1.15

$$\text{for any } i \in \{1, \ldots, n\} \; E_i(x,y) := F_i(x,y) \vee F_i(y,x)$$

is an equivalence relation which partitions $p(M)$ on convex classes. By Fact 1.4 E_i partitions $p(M)$ on infinitely many classes, so that the induced order on classes is a dense order without endpoints. Item 2 follows from Fact 2.5. It is remained to prove item 3.
First show the following holds:

$$\text{for any } \alpha_1,\ \alpha_2,\ \alpha_3 \in E_1(M,\alpha) \text{ such that } \alpha_1 < \alpha_2, \alpha_1 < \alpha_3$$

$$tp(\langle \alpha_1, \alpha_2 \rangle / A) = tp(\langle \alpha_1, \alpha_3 \rangle / A) \quad (*)$$

For a contradiction, suppose that there are $\alpha_1, \alpha_2, \alpha_3 \in E_1(M,\alpha)$ such that $\alpha_1 < \alpha_2$, $\alpha_1 < \alpha_3$ and there is an A-definable formula $\phi(x_1, x_2)$ such that

$$M \models \phi(\alpha_1, \alpha_2) \wedge \neg\phi(\alpha_1, \alpha_3)$$

By weak o-minimality of M $\phi(\alpha_1, M)$ is a union of finitely many convex sets. Let $\phi_0(\alpha_1, x)$ be a maximal convex subformula of $\phi(\alpha_1, x)$ such that $M \models \phi_0(\alpha_1, \alpha_2)$.

If $\alpha_2 < \alpha_3$ then consider the following formula:

$$F(x, \alpha_1) := \alpha_1 \leq x \wedge \forall y[\phi_0(\alpha_1, y) \to x \leq y]$$

It is clear that $M \models F(\alpha_2, \alpha_1) \wedge \neg F(\alpha_3, \alpha_1)$. It can understand that $F(x, y)$ is a non-trivial p-stable convex to the right formula so that $F(M, \alpha_1) \subset F_1(M, \alpha_1)$. This contradicts to the hypothesis that $F_1(x, y)$ is minimal non-trivial. Thus, $(*)$ is proved.

Similarly, it can show that

$$\text{for any } \alpha_1,\ \alpha_2,\ \alpha_3 \in E_1(M, \alpha) \text{ such that } \alpha_1 > \alpha_2,\ \alpha_1 > \alpha_3$$
$$tp(\langle\alpha_1, \alpha_2\rangle/A) = tp(\langle\alpha_1, \alpha_3\rangle/A) \quad (**)$$

Proof of $(**)$ is analogous to the proof of $(*)$.

Now consider arbitrary tuples $\langle\alpha_1, \alpha_2\rangle$, $\langle\beta_1, \beta_2\rangle$ of increasing elements from $E_1(M, \alpha)$ and we prove their types over A coincide.

Case 1. $\alpha_1 < \beta_2$.

Then according to $(*), (**)$ we have that for any A-definable formula $\phi(x, y)$

$$M \models \phi(\alpha_1, \alpha_2) \Leftrightarrow M \models \phi(\alpha_1, \beta_2) \Leftrightarrow M \models \phi(\beta_1, \beta_2)$$

Case 2. $\alpha_1 = \beta_2$.

Then according to $(*), (**)$ we have that for any A-definable formula $\phi(x, y)$

$$M \models \phi(\alpha_1, \alpha_2) \Leftrightarrow M \models \phi(\beta_1, \alpha_2) \Leftrightarrow M \models \phi(\beta_1, \beta_2)$$

$\square$

Corollary 2.10 *Let T be an $\aleph_0$-categorical o-minimal theory. Then for any $M \models T$ for any finite $A \subseteq M$ for any non-algebraic type $p \in S_1(A)$ $p(M)$ is indiscernible over A.*

3 Indiscernibility of the set of realizations of an one-type

In [5] we have introduced the following notion: rank of convexity of a formula with one free variable.

Definition 3.1 [5] Let T be a weakly o-minimal theory, M be a sufficiently saturated model of T, and let $\phi(x, \bar{a}), \bar{a} \in M$ be an arbitrary formula with one free variable.

$RC(\phi(x, \bar{a}))$ is defined as follows:

1) $RC(\phi(x, \bar{a})) = -1$ if $M \models \neg\exists x \phi(x, \bar{a})$.

2) $RC(\phi(x, \bar{a})) \geq 0$ if $M \models \exists x \phi(x, \bar{a})$.

3) $RC(\phi(x, \bar{a})) \geq 1$ if $|\phi(M, \bar{a})| \geq \omega$.

4) $RC(\phi(x, \bar{a})) \geq \alpha + 1$ if there is a parametrically definable equivalence relation $E(x, y)$ such that there are $b_i, i \in \omega$ which satisfy the following:

- For every $i, j \in \omega$, whenever $i \neq j$ then $M \models \neg E(b_i, b_j)$
- For every $i \in \omega$ $RC(\phi(x, \bar{a}) \& E(x, b_i)) \geq \alpha$
- For every $i \in \omega$ $E(M, b_i)$ is a convex subset of $\phi(M, \bar{a})$

5) $RC(\phi(x, \bar{a})) \geq \delta$ if $RC(\phi(x, \bar{a})) \geq \alpha$ for all $\alpha \leq \delta$ (δ is limit)

If $RC(\phi(x, \bar{a})) = \alpha$ for some α we say that $RC(\phi(x, \bar{a}))$ is defined. Otherwise (i.e. if $RC(\phi(x, \bar{a})) \geq \alpha$, for all α) we put $RC(\phi(x, \bar{a})) = \infty$.

Fact 3.2 *Let T be a weakly o-minimal theory of convexity rank 1, $M \models T$, $A \subseteq M$. Then any quasisolitary type over A is solitary.*

Lemma 3.3 *Let T be an $\aleph_0$-categorical weakly o-minimal theory, $M \models T$, $A \subseteq M$, $p \in S_1(A)$, M be $|A|^+$-saturated. Then for any p-stable convex to the right (left) formula $F(x, y)$ there is a p-stable convex to the right (left) formula $F_1(x, y)$ such that*

1) For any $\alpha \in p(M)$ $F(M, \alpha) \subseteq F_1(M, \alpha)$.

2) $F_1(x, y)$ is locally p-constant.

Proof of Lemma 3.3.

If p is quasisolitary then it is nothing to prove. Suppose that p is not quasisolitary, and let $F(x, y, \bar{a}), \bar{a} \in A$ be a non-trivial p-stable convex to the right formula. If it is locally p-constant then it is nothing to prove. Suppose that it is not locally p-constant. Then only the following cases are possible:

Case 1. There are $\alpha, \beta \in p(M)$ such that

$$M \models \exists x[F(\beta, \alpha, \bar{a}) \wedge F(x, \beta, \bar{a}) \wedge \neg F(x, \alpha, \bar{a})]$$

Case 2. There are $\alpha, \beta \in p(M)$ such that

$$M \models \exists x[F(\beta,\alpha,\bar{a}) \wedge F(x,\alpha,\bar{a}) \wedge \neg F(x,\beta,\bar{a}) \wedge x > \beta]$$

Case 1. Consider the following formula:

$$F'(x,y,\bar{a}) := \exists t[F(t,y,\bar{a}) \wedge F(x,t,\bar{a})]$$

It is clear that for any $\alpha \in p(M)$ $F(M,\alpha,\bar{a}) \subset F'(M,\alpha,\bar{a})$. By Lemma 1.14 we have this formula is p-stable convex to the right. If $F'(x,y,\bar{a})$ is locally p-constant then we are done. If not then we return either to case 1 or to case 2.

Case 2. In this case there is $\gamma \in p(M)$ such that

$$M \models F(\gamma,\alpha,\bar{a}) \wedge \neg F(\gamma,\beta,\bar{a}) \wedge \gamma > \beta$$

Let $F_0(\beta, M, \bar{a})$ be a maximal convex $\bar{a}, \beta$-definable subset of $F(\beta, M, \bar{a})$ such that $\alpha \in F_0(\beta, M, \bar{a})$.
Consider the following formula:

$$F_1(x,y,\bar{a}) := \exists z[F_0(y,z,\bar{a}) \wedge F(x,z,\bar{a}) \wedge x \geq y]$$

It is clear that for any $\alpha \in p(M)$ $F(M,\alpha,\bar{a}) \subset F_1(M,\alpha,\bar{a})$. By Lemma 1.11 we have this formula is p-stable convex to the right and it is not locally p-decreasing. If $F_1(x,y,\bar{a})$ is locally p-constant then we are done. If not then we return to case 1.
This process must be finished in finitely many steps, otherwise we will have infinitely many non-equivalent $\bar{a}$-definable formulas with two quantifier-free variables, contradicting to $\aleph_0$-categoricity of T. □

Lemma 3.4 *Let T be an $\aleph_0$-categorical weakly o-minimal theory of finite convexity rank. Then for any $M \models T$ for any $A \subseteq M$ every non-algebraic type $p \in S_1(A)$ is quasisolitary.*

Proof of Lemma 3.4.
For a contradiction, suppose that there exist $A \subseteq M$ and a non-algebraic type $p \in S_1(A)$ which is not quasisolitary.
Step 1. Consider an arbitrary non-trivial p-stable convex to the right formula $F_1(x,y,\bar{a}_1)$, where $\bar{a}_1 \in A$. By Lemma 3.3 there is a p-stable convex to the right formula $F_1'(x,y,\bar{a}_1)$ such that

1) For any $\alpha \in p(M)$ $F_1(M,\alpha,\bar{a}_1) \subseteq F_1'(M,\alpha,\bar{a}_1)$.
2) $F_1'(x,y,\bar{a}_1)$ is locally p-constant.

Then by Lemma 1.15 there is an A-definable equivalence relation $E_1(x,y)$ which partitions $p(M)$ on convex classes. By Fact 2.4 these classes are densely ordered without endpoints. Consequently, $RC(x=x) \geq 2$.
Step 2. As p is not quasisolitary then there is a p-stable convex to the right formula $F_2(x,y,\bar{a}_2), \bar{a}_2 \in A$ which is greater than $F_1'(x,y,\bar{a}_1)$. By Lemma 3.3 there is a p-stable convex to the right formula $F_2'(x,y,\bar{a}_2)$ such that

1) For any $\alpha \in p(M)$ $F_2(M,\alpha,\bar{a}_2) \subseteq F_2'(M,\alpha,\bar{a}_2)$.
2) $F_2'(x,y,\bar{a}_2)$ is locally p-constant.

Then by Lemma 1.15 there is an A-definable equivalence relation $E_2(x,y)$ which partitions $p(M)$ on convex classes. By Fact 2.4 these classes are densely ordered without endpoints, and by Fact 2.5 E_1 partitions each E_2-class on infinitely many E_1-classes. Consequently, $RC(x=x) \geq 3$.
As p is not quasisolitary then we can continue this procedure infinitely many steps, but in ω steps we have that $RC(x=x) \geq \omega$. This contradicts to hypothesis of the lemma. □

Corollary 3.5 *Let T be an $\aleph_0$-categorical weakly o-minimal theory of convexity rank 1. Then for any $M \models T$ for any $A \subseteq M$ every non-algebraic type $p \in S_1(A)$ is solitary.*

Lemma 3.6 *Let T be an $\aleph_0$-categorical weakly o-minimal theory of convexity rank 1. Then for any $M \models T$ for any $A \subseteq M$ for any non-algebraic type $p \in S_1(A)$ $p(M)$ is indiscernible over A.*

Proof of Lemma 3.6.
Let $A \subseteq M$, $p \in S_1(A)$ be non-algebraic. Show that $p(M)$ is indiscernible over A.
Step 1. $p(M)$ is 1-indiscernible over A.
It is obvious by the definition of $p(M)$.
Step 2. $p(M)$ is 2-indiscernible over A.
This follows from Lemma 1.20.

Step n+1. Suppose that $p(M)$ is n-indiscernible over A. Prove that $p(M)$ n+1-indiscernible over A.
First show the following holds:

$$\text{for any } \bar{\alpha} = \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle, \gamma_1, \gamma_2 \in p(M) \text{ such that}$$

$$\alpha_1 < \alpha_2 < \ldots < \alpha_n, \alpha_n < \gamma_1, \alpha_n < \gamma_2$$

$$tp(\bar{\alpha}, \gamma_1/A) = tp(\bar{\alpha}, \gamma_2/A) \quad (1)_+$$

For a contradiction, suppose that:

$$\text{there exist } \bar{\alpha} = \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle, \gamma_1, \gamma_2 \in p(M) \text{ such that}$$

$$\alpha_1 < \alpha_2 < \ldots < \alpha_n, \ \alpha_n < \gamma_1, \ \alpha_n < \gamma_2 \text{ and}$$

$$tp(\bar{\alpha}, \gamma_1/A) \neq tp(\bar{\alpha}, \gamma_2/A).$$

Then there is an A-definable formula $\phi(x_1, x_2, \ldots, x_n, y)$ such that

$$M \models \phi(\alpha, \gamma_1) \wedge \neg\phi(\bar{\alpha}, \gamma_2).$$

As $p(M)$ is n-indiscernible over A then $p(M) \cap \{a \in M \mid a > \alpha_{n-1}\}$ is 1-indiscernible over $A \cup \{\bar{\alpha} \setminus \{\alpha_n\}\}$. Consequently, there is a type $p' \in S_1(A \cup \{\bar{\alpha} \setminus \{\alpha_n\}\})$ such that

$$p'(M) = p(M) \bigcap \{a \in M \mid a > \alpha_{n-1}\}$$

By Corollary 3.5 the type p' is solitary.
By weak o-minimality of M there is a convex subformula $\phi_0(\bar{\alpha}, x)$ of $\phi(\bar{\alpha}, x)$ such that $\gamma_1 \in \phi_0(\bar{\alpha}, M)$.
If $\gamma_1 < \gamma_2$ then consider the following formula:

$$F(x, \alpha_n, \bar{\alpha} \setminus \{\alpha_n\}) := \alpha_n \leq x \wedge \forall y[\phi_0(\bar{\alpha} \setminus \{\alpha_n\}, \alpha_n, y) \rightarrow x \leq y]$$

It is clear that $M \models F(\gamma_1, \alpha_n, \bar{\alpha} \setminus \{\alpha_n\}) \wedge \neg F(\gamma_2, \alpha_n, \bar{\alpha} \setminus \{\alpha_n\})$. It can understand that $F(x, y, \bar{\alpha} \setminus \{\alpha_n\})$ is a non-trivial p'-stable convex to the right formula. This contradicts to a solitarity of p'.
If $\gamma_1 > \gamma_2$ then consider the following formula:

$$F(x, \alpha_n, \bar{\alpha} \setminus \{\alpha_n\}) := \alpha_n \leq x \wedge \forall y[\phi_0(\bar{\alpha} \setminus \{\alpha_n\}, \alpha_n, y) \rightarrow x < y]$$

It is clear that $M \models F(\gamma_2, \alpha_n, \bar{\alpha} \setminus \{\alpha_n\}) \wedge \neg F(\gamma_1, \alpha_n, \bar{\alpha} \setminus \{\alpha_n\})$. It can understand that $F(x, y, \bar{\alpha} \setminus \{\alpha_n\})$ is a non-trivial p'-stable convex to the right formula. This contradicts to a solitarity of p'.
Thus, $(1)_+$ is proved.
Similarly it can show that the following holds:

$$\text{for any } \bar{\alpha} = \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle, \gamma_1, \gamma_2 \in p(M) \text{ such that}$$

$$\alpha_1 > \alpha_2 > \ldots > \alpha_n, \ \alpha_n > \gamma_1, \ \alpha_n > \gamma_2$$

$$tp(\bar{\alpha}, \gamma_1/A) = tp(\bar{\alpha}, \gamma_2/A) \quad (1)_-$$

Now show that for any $i \in \{1, \ldots, n-1\}$ the following holds:

$$\text{for any } \bar{\alpha} = \langle \alpha_1, \ldots, \alpha_n \rangle, \gamma_1, \gamma_2 \in p(M) \text{ such that}$$

$$\alpha_1 < \ldots < \alpha_n, \ \alpha_i < \gamma_1 < \alpha_{i+1}, \ \alpha_i < \gamma_2 < \alpha_{i+1}$$

$$tp(\bar{\alpha}, \gamma_1/A) = tp(\bar{\alpha}, \gamma_2/A) \quad (1)_i$$

As $p(M)$ is n-indiscernible over A then for $i \geq 2$ the following set $p(M) \cap \{a \in M \mid \alpha_{i-1} < a < \alpha_{i+1}\}$ is 1-indiscernible over $A \cup \{\bar{\alpha} \setminus \{\alpha_i\}\}$. Consequently, there is a type $p' \in S_1(A \cup \{\bar{\alpha} \setminus \{\alpha_i\}\})$ such that

$$p'(M) = p(M) \bigcap \{a \in M \mid \alpha_{i-1} < a < \alpha_{i+1}\}$$

For $i = 1$ $p(M) \cap \{a \in M \mid \alpha_1 < a < \alpha_3\}$ is 1-indiscernible over $A \cup \{\bar{\alpha} \setminus \{\alpha_2\}\}$. Consequently, there is a type $p'' \in S_1(A \cup \{\bar{\alpha} \setminus \{\alpha_2\}\})$ such that

$$p''(M) = p(M) \bigcap \{a \in M \mid \alpha_1 < a < \alpha_3\}$$

If we assume $(1)_i$ does not hold then either a non-trivial p'-stable or a non-trivial p''-stable convex to the right formula arises contradicting to a solitarity of p' or p''.
Now show that

$$\text{for any } \bar{\alpha} = \langle \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \rangle, \beta_1, \beta_2, \gamma_1, \gamma_2 \in p(M) \text{ such that}$$

$$\alpha_1 < \alpha_2 < \ldots < \alpha_{n-1}, \alpha_{n-1} < \beta_1 < \beta_2, \alpha_{n-1} < \gamma_1 < \gamma_2$$

$$tp(\bar{\alpha}, \beta_1, \beta_2/A) = tp(\bar{\alpha}, \gamma_1, \gamma_2/A) \quad (2)_+$$

Case 1. $\beta_1 < \gamma_2$.
Then by $(1)_+, (1)_{n-1}$ for any A-definable formula $\phi(x_1, \ldots, x_{n-1}, y, z)$ we have

$$M \models \phi(\bar{\alpha}, \beta_1, \beta_2) \Leftrightarrow M \models \phi(\bar{\alpha}, \beta_1, \gamma_2) \Leftrightarrow M \models \phi(\bar{\alpha}, \gamma_1, \gamma_2).$$

Case 2. $\beta_1 \geq \gamma_2$.
Then by $(1)_+, (1)_{n-1}$ for any A-definable formula $\phi(x_1, \ldots, x_{n-1}, y, z)$ we have

$$M \models \phi(\bar{\alpha}, \gamma_1, \gamma_2) \Leftrightarrow M \models \phi(\bar{\alpha}, \gamma_1, \beta_2) \Leftrightarrow M \models \phi(\bar{\alpha}, \beta_1, \beta_2).$$

Similarly, it can show that

$$\text{for any } \bar{\alpha} = \langle \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \rangle, \beta_1, \beta_2, \gamma_1, \gamma_2 \in p(M) \text{ such that}$$
$$\alpha_1 > \alpha_2 > \ldots > \alpha_{n-1},\ \alpha_{n-1} > \beta_1 > \beta_2,\ \alpha_{n-1} > \gamma_1 > \gamma_2$$
$$tp(\bar{\alpha}, \beta_1, \beta_2/A) = tp(\bar{\alpha}, \gamma_1, \gamma_2/A) \quad (2)_-$$

Now show that for any $i \in \{1, \ldots, n-2\}$

$$\text{for any } \bar{\alpha} = \langle \alpha_1, \alpha_2, \ldots, \alpha_{n-1} \rangle, \beta_1, \beta_2, \gamma_1, \gamma_2 \in p(M)$$
$$\text{such that } \alpha_1 < \alpha_2 < \ldots < \alpha_{n-1},\ \alpha_i < \beta_1 < \beta_2 < \alpha_{i+1},$$
$$\alpha_i < \gamma_1 < \gamma_2 < \alpha_{i+1}$$
$$tp(\bar{\alpha}, \beta_1, \beta_2/A) = tp(\bar{\alpha}, \gamma_1, \gamma_2/A) \quad (2)_i$$

Case 1. $\beta_1 < \gamma_2$.
Then by $(1)_{i+1}, (1)_i$ for any A-definable formula $\phi(x_1, \ldots, x_{n-1}, y, z)$ we have

$$M \models \phi(\bar{\alpha}, \beta_1, \beta_2) \Leftrightarrow M \models \phi(\bar{\alpha}, \beta_1, \gamma_2) \Leftrightarrow M \models \phi(\bar{\alpha}, \gamma_1, \gamma_2).$$

Case 2. $\beta_1 \geq \gamma_2$.
Then by $(1)_{i+1}, (1)_i$ for any A-definable formula $\phi(x_1, \ldots, x_{n-1}, y, z)$ we have

$$M \models \phi(\bar{\alpha}, \gamma_1, \gamma_2) \Leftrightarrow M \models \phi(\bar{\alpha}, \gamma_1, \beta_2) \Leftrightarrow M \models \phi(\bar{\alpha}, \beta_1, \beta_2).$$

Thus, for any $i \in \{1, \ldots, n-2\}$ $(2)_i$ is proved.
Now suppose that for any $j \in \{1, \ldots, k\}$, where $2 \leq k < n$, the following properties hold:

$$\text{for any } \bar{\alpha} = \langle \alpha_1, \ldots, \alpha_{n+1-j} \rangle, \beta_1, \ldots, \beta_j, \gamma_1, \ldots, \gamma_j \in p(M)$$

$$\text{such that } \alpha_1 < \ldots < \alpha_{n+1-j},\ \beta_1 < \ldots < \beta_j,$$
$$\gamma_1 < \ldots < \gamma_j,\ \alpha_{n+1-j} < \beta_1, \alpha_{n+1-j} < \gamma_1$$
$$tp(\bar{\alpha}, \bar{\beta}/A) = tp(\bar{\alpha}, \bar{\gamma}/A) \quad (j)_+$$
$$\text{for any } \bar{\alpha} = \langle \alpha_1, \ldots, \alpha_{n+1-j} \rangle, \beta_1, \ldots, \beta_j, \gamma_1, \ldots, \gamma_j \in p(M)$$
$$\text{such that } \alpha_1 > \ldots > \alpha_{n+1-j},\ \beta_1 > \ldots > \beta_j,$$
$$\gamma_1 > \ldots > \gamma_j,\ \alpha_{n+1-j} > \beta_1, \alpha_{n+1-j} > \gamma_1$$
$$tp(\bar{\alpha}, \bar{\beta}/A) = tp(\bar{\alpha}, \bar{\gamma}/A) \quad (j)_-$$

For every $i \in \{1, \ldots, n-j\}$

$$\text{for any } \bar{\alpha} = \langle \alpha_1, \ldots, \alpha_{n+1-j} \rangle, \bar{\beta} = \langle \beta_1, \ldots, \beta_j \rangle,$$
$$\bar{\gamma} = \langle \gamma_1, \ldots, \gamma_j \rangle \in p(M), \text{ such that } \alpha_1 < \ldots < \alpha_{n+1-j},$$
$$\alpha_i < \beta_1 < \ldots < \beta_j < \alpha_{i+1}, \alpha_i < \gamma_1 < \ldots < \gamma_j < \alpha_{i+1}$$
$$tp(\bar{\alpha}, \bar{\beta}/A) = tp(\bar{\alpha}, \bar{\gamma}/A) \quad (j)_i$$

Show that the following holds:

$$\text{for any } \bar{\alpha} = \langle \alpha_1, \ldots, \alpha_{n-j} \rangle, \bar{\beta} = \langle \beta_1, \ldots, \beta_{j+1} \rangle,$$
$$\bar{\gamma} = \langle \gamma_1, \ldots, \gamma_{j+1} \rangle \in p(M), \text{ such that } \alpha_1 < \ldots < \alpha_{n-j},$$
$$\beta_1 < \ldots < \beta_{j+1},\ \gamma_1 < \ldots < \gamma_{j+1},\ \alpha_{n-j} < \beta_1,\ \alpha_{n-j} < \gamma_1$$
$$tp(\bar{\alpha}, \bar{\beta}/A) = tp(\bar{\alpha}, \bar{\gamma}/A) \quad (j+1)_+$$

Case 1. $\beta_1 < \gamma_2$.
Then by $(j)_+, (1)_{n-j}$ for any A-definable formula $\phi(x_1, .., x_{n-j}, y_1, .., y_{j+1})$ we have

$$M \models \phi(\bar{\alpha}, \beta_1, \beta_2, \ldots, \beta_{j+1}) \Leftrightarrow M \models \phi(\bar{\alpha}, \beta_1, \gamma_2, \ldots, \gamma_{j+1}),$$
$$M \models \phi(\bar{\alpha}, \beta_1, \gamma_2, \ldots, \gamma_{j+1}) \Leftrightarrow M \models \phi(\bar{\alpha}, \gamma_1, \gamma_2, \ldots, \gamma_{j+1}).$$

Case 2. $\beta_1 \geq \gamma_2$.
Then by $(j-1)_+, (2)_{n-j}$ for any A-definable formula $\phi(x_1, .., x_{n-j}, y_1, .., y_{j+1})$ we have

$$M \models \phi(\bar{\alpha}, \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{j+1}) \Leftrightarrow M \models \phi(\bar{\alpha}, \gamma_1, \gamma_2, \beta_3, \ldots, \beta_{j+1}),$$

$$M \models \phi(\bar\alpha, \gamma_1, \gamma_2, \beta_3, \dots, \beta_{j+1}) \Leftrightarrow M \models \phi(\bar\alpha, \beta_1, \beta_2, \beta_3, \dots, \beta_{j+1}).$$

Similarly we can show that

$$\text{for any } \bar\alpha = \langle \alpha_1, \dots, \alpha_{n-j} \rangle,\ \bar\beta = \langle \beta_1, \dots, \beta_{j+1} \rangle,$$
$$\bar\gamma = \langle \gamma_1, \dots, \gamma_{j+1} \rangle \in p(M), \text{ such that } \alpha_1 > \dots > \alpha_{n-j},$$
$$\beta_1 > \dots > \beta_{j+1},\ \gamma_1 > \dots > \gamma_{j+1},\ \alpha_{n-j} > \beta_1,\ \alpha_{n-j} > \gamma_1$$
$$tp(\bar\alpha, \bar\beta/A) = tp(\bar\alpha, \bar\gamma/A) \quad (j+1)_-$$

Now show that for any $i \in \{1, \dots, n-j-1\}$

$$\text{for any } \bar\alpha = \langle \alpha_1, \dots, \alpha_{n-j} \rangle,\ \bar\beta = \langle \beta_1, \dots, \beta_{j+1} \rangle,$$
$$\bar\gamma = \langle \gamma_1, \dots, \gamma_{j+1} \rangle \in p(M), \text{ such that } \alpha_1 < \dots < \alpha_{n-j},$$
$$\alpha_i < \beta_1 < \dots < \beta_{j+1} < \alpha_{i+1},\ \alpha_i < \gamma_1 < \dots < \gamma_{j+1} < \alpha_{i+1}$$
$$tp(\bar\alpha, \bar\beta/A) = tp(\bar\alpha, \bar\gamma/A) \quad (j+1)_i$$

Case 1. $\beta_1 < \gamma_2$.
Then by $(j)_{i+1}, (1)_i$ for any A-definable formula $\phi(x_1, .., x_{n-j}, y_1, .., y_{j+1})$ we have

$$M \models \phi(\bar\alpha, \beta_1, \beta_2, \dots, \beta_{j+1}) \Leftrightarrow M \models \phi(\bar\alpha, \beta_1, \gamma_2, \dots, \gamma_{j+1}),$$
$$M \models \phi(\bar\alpha, \beta_1, \gamma_2, \dots, \gamma_{j+1}) \Leftrightarrow M \models \phi(\bar\alpha, \gamma_1, \gamma_2, \dots, \gamma_{j+1}).$$

Case 2. $\beta_1 \geq \gamma_2$.
Then by $(j-1)_{i+2}, (2)_i$ for any A-definable formula $\phi(x_1, .., x_{n-j}, y_1, .., y_{j+1})$ we have

$$M \models \phi(\bar\alpha, \gamma_1, \gamma_2, \gamma_3 \dots, \gamma_{j+1}) \Leftrightarrow M \models \phi(\bar\alpha, \gamma_1, \gamma_2, \beta_3, \dots, \beta_{j+1}),$$
$$M \models \phi(\bar\alpha, \gamma_1, \gamma_2, \beta_3, \dots, \beta_{j+1}) \Leftrightarrow M \models \phi(\bar\alpha, \beta_1, \beta_2, \beta_3, \dots, \beta_{j+1}).$$

Thus, for every $i \in \{1, \dots, n-j-1\}$ $(j+1)_i$ is proved.
Now show that

$$\text{for any } \bar\alpha = \langle \alpha_1, \alpha_2, \dots, \alpha_{n+1} \rangle, \bar\beta = \langle \beta_1, \beta_2, \dots, \beta_{n+1} \rangle \in p(M)$$
$$\text{such that } \alpha_1 < \alpha_2 < \dots < \alpha_{n+1},\ \beta_1 < \beta_2 < \dots < \beta_{n+1}$$
$$tp(\bar\alpha/A) = tp(\bar\beta/A)$$

Case 1. $\alpha_1 < \beta_2$.
Then by $(n)_+, (1)_-$ for any A-definable formula $\phi(x_1, x_2, \dots, x_{n+1})$ we have

$$M \models \phi(\alpha_1, \alpha_2, \dots, \alpha_{n+1}) \Leftrightarrow M \models \phi(\alpha_1, \beta_2, \dots, \beta_{n+1}),$$
$$M \models \phi(\alpha_1, \beta_2, \dots, \beta_{n+1}) \Leftrightarrow M \models \phi(\beta_1, \beta_2, \dots, \beta_{n+1}).$$

Case 2. $\alpha_1 \geq \beta_2$.
Then by $(n-1)_+, (2)_-$ for any A-definable formula $\phi(x_1, x_2, \dots, x_{n+1})$ we have

$$M \models \phi(\beta_1, \beta_2, \beta_3, \dots, \beta_{n+1}) \Leftrightarrow M \models \phi(\beta_1, \beta_2, \alpha_3, \dots, \alpha_{n+1}),$$
$$M \models \phi(\beta_1, \beta_2, \alpha_3, \dots, \alpha_{n+1}) \Leftrightarrow M \models \phi(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{n+1}).$$

Consequently, $p(M)$ is n+1-indiscernible over A. □

Corollary 3.7 *Let T be an $\aleph_0$-categorical o-minimal theory. Then for any $M \models T$ for any $A \subseteq M$ for any non-algebraic type $p \in S_1(A)$ $p(M)$ is indiscernible over A.*

Corollary 3.8 *Let T be an $\aleph_0$-categorical weakly o-minimal theory. Then the following conditions are equivalent:*
(1) $RC(x = x) = 1$.
(2) For any $M \models T$ for any $A \subseteq M$ every non-algebraic type $p \in S_1(A)$ is solitary.
(3) For any $M \models T$ for any $A \subseteq M$ for any non-algebraic type $p \in S_1(A)$ $p(M)$ is indiscernible over A.

Proof of Corollary 3.8.
(1) ⇒ (2) follows from Corollary 3.5.
(2) ⇒ (3) follows from Lemma 3.6.
(3) ⇒ (1). For a contradiction, suppose that $RC(x = x) > 1$. Then there exist $M \models T$, an equivalence relation $E(x, y, \bar a), \bar a \in M$ which partitions M on infinitely many infinite convex classes. As T is $\aleph_0$-categorical then there is only finitely many one-types over $\{\bar a\}$ and all of them are principal. Then there is at least one non-algebraic type $p \in S_1(\{\bar a\})$ such that $p(M)$ is partitioned on infinitely many infinite convex classes. It is obvious that $p(M)$ is not 2 indiscernible over $\{\bar a\}$. This contradicts to the hypothesis. □

Definition 3.9 [3] A complete theory T is said to be *quasibinary* if for an model M of T for any $A, B \subseteq M$

$$[B \text{ is 2-indiscernible over } A \Rightarrow B \text{ is indiscernible over } A]$$

Corollary 3.10 *Any $\aleph_0$-categorical weakly o-minimal theory of convexi rank 1 is quasibinary.*

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МОДАЛЬНЫЕ П-СХЕМЫ

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В этой статье изучаются модальные пропозициональные П-схемы. В [1] было доказано, что для любой позитивной П-схемы существует П-формула, определяющая наименьшее решение этой схемы в любой частично-упорядоченной модели Крипке с обрывом возрастающих цепей. Здесь мы представим более широкий класс моделей, на которых определимы наименьшие решения позитивных П-схем. Этот класс включает в себя линейные модели, основанные на натуральных или целых числах. Однако, определимость здесь немного другая. Если в частично-упорядоченных моделях с обрывом возрастающих цепей достаточно было одной П-формулы, то здесь потребуется конечное множество П-формул.

Что касается иррефлексивного случая, то наименьшие решения позитивных П-схем определимы П-формулой в строго частично-упорядоченных моделях с обрывом возрастающих цепей. Этот результат можно вывести из теоремы о неподвижной точке [3], либо получить применением конструкции из [2]. В моделях из рассматриваемого в данной статье класса могут нарушаться как единственность, так и существование решений модализованных схем, т.е. на них не переносится теорема о неподвижной точке. Поэтому в доказательстве применяем конструкцию из [2].

Эти результаты имеют следствия для логик.

Модальные пропозициональные формулы составляются из констант $\perp$, $\top$ и пропозициональных переменных $p, q, r, \ldots$ с помощью связок $\wedge$, $\vee$, $\neg$, $\Box$.

Шкала Крипке $< W, R >$ состоит из непустого множества W и бинарного отношения R на W. Модель Крипке $< W, R, v >$ состоит из шкалы Крипке $< W, R >$ и означивания v. Означивание v - это функция, которая

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каждой переменной q ставит в соответствие подмножество $v(q)$ множества W. Это подмножество $v(q)$ назовем значением переменной q и будем обозначать соответствующей прописной буквой Q. Функция v естественным образом продолжается на формулы: константе $\bot$ всегда соответствует пустое множество, константе $\top$ - множество W, связкам $\wedge$, $\vee$, $\neg$, $\blacktriangle$, Σ соответствуют дополнение, пересечение, обьединение множеств и следующие операции на множествах: $\blacktriangle A = \{x|\forall y(xRy \Rightarrow y \in A)\}$, $\Sigma A = \{x|\exists y(xRy \wedge y \in A)\}$. Таким образом, значение формулы всегда является подмножеством множества W. Значение формулы $\alpha(q_1,\ldots,q_n)$ будем обозначать через $\alpha(Q_1,\ldots,Q_n)$.

Импликацию и эквивалентность будем обозначать $\Rightarrow$ и $\equiv$. Еще введем сокращения $\Gamma\alpha = \alpha \wedge \blacktriangle\alpha$ и $\exists\alpha = \alpha \vee \Sigma\alpha$.

Если элемент модели x принадлежит значению формулы α, то будем это обозначать $x \models \alpha$ и говорить, что α истинна на элементе x. Если x - не принадлежит, то обозначаем $x \not\models \alpha$ и говорим, что α ложна или опровергается на x.

Рассмотрим модель Крипке $< W, R, v >$ в которой заданы значения $Q_1,\ldots,Q_n$ переменных $q_1,\ldots,q_n$, а значение переменной p не задано. Пусть дана формула $\varphi(p,q_1,\ldots,q_n)$ от переменных $p,q_1,\ldots,q_n$. Модальной пропозициональной схемой назовем уравнение $P = \varphi(P,Q_1,\ldots,Q_n)$. Решением схемы назовем подмножество P множества W, при котором выполняется равенство множеств P и $\varphi(P,Q_1,\ldots,Q_n)$. Другими словами, мы ищем значение переменной p, при котором значения p и $\varphi(p,q_1,\ldots,q_n)$ совпадают. Заметим, что решение - это неподвижная точка оператора, сопоставляющего множеству P множество $\varphi(P,Q_1,\ldots,Q_n)$. Решение P назовем наименьшим решением, если для любого другого решения P' выполняется $P \subseteq P'$. Если решение P совпадает в модели со значением некоторой формулы $\omega(q_1,\ldots,q_n)$, то говорим, что эта формула определяет решение P в данной модели.

Введем некоторые классы схем. Если переменная p входит в формулу $\varphi(p,q_1,\ldots,q_n)$ только позитивно, т.е. каждое вхождение находится под действием четного числа отрицаний, то называем схему $P = \varphi(P,Q_1,\ldots,Q_n)$ позитивной.

Модальной П-формулой назовем формулу, составленную с помощью связок $\wedge$, $\vee$, $\blacktriangle$ из формул, не содержащих модальностей. Если формула $\varphi(p,q_1,\ldots,q_n)$ является П-формулой, то схему называем П-схемой.

Несколько упрощает дело следующий общий результат, показывающий, что мы можем ограничиться формулами, в которых все вхождения переменной p модализованы, т.е. находятся под действием модальности.

Рассмотрим позитивную схему $P = \varphi(P,Q_1,\ldots,Q_n)$. В формуле $\varphi(p,q_1,\ldots,q_n)$ заменим на $\bot$ все вхождения p, не находящиеся под действием модальности. Полученную формулу обозначим $\zeta(p,q_1,\ldots,q_n)$. Ясно, что схема $P = \zeta(P,Q_1,\ldots,Q_n)$ - позитивная.

ПРЕДЛОЖЕНИЕ 1. Наименьшие решения схем $P = \varphi(P,Q_1,\ldots,Q_n)$ и $P = \zeta(P,Q_1,\ldots,Q_n)$ совпадают на любой модели Крипке $< W, R, v >$.

ДОКАЗАТЕЛЬСТВО. Ясно, что всегда $\zeta(P,Q_1,\ldots,Q_n) \subseteq \varphi(P,Q_1,\ldots,Q_n)$. Пусть P - наименьшее решение схемы $P = \varphi(P,Q_1,\ldots,Q_n)$. Тогда $\zeta(P,Q_1,\ldots,Q_n) \subseteq P$. Поэтому наименьшее решение схемы $P = \zeta(P,Q_1,\ldots,Q_n)$ содержится в P. Пусть теперь P' - наименьшее решение схемы $P = \zeta(P,Q_1,\ldots,Q_n)$. При обратной замене $\bot$ на P' значение формулы может увеличиться только внутри P'. Так как $\zeta(P',Q_1,\ldots,Q_n)$ совпадает с P', то значение не изменится. Поэтому $P' = \varphi(P',Q_1,\ldots,Q_n)$. Следовательно, P' - решение схемы $P = \varphi(P,Q_1,\ldots,Q_n)$ и наименьшее решение этой схемы содержится в P'. Итак, $P = P'$. Предложение доказано.

Подмножество A шкалы $< W, R >$ называем конфинальным, если для любого $x \in W$ найдется $y \in A$ такой, что xRy.

Зададим класс C частично-упорядоченных шкал следующим определением.

Шкала $< W, \leq >$ лежит в классе C, если любая бесконечная возрастающая цепь $x_1 < x_2 < \ldots$ является конфинальным множеством в $< W, \leq >$.

В частности, шкала в которой нет бесконечных возрастающих цепей лежит в C.

Шкалы из класса C будем еще называть шкалами с условием конфинальности бесконечных возрастающих цепей.

Класс CM моделей Крипке состоит из моделей $< W, \leq, v >$ таких, что шкала $< W, \leq >$ лежит в C.

Натуральные и целые числа с естественным порядком являются шкалами с условием конфинальности бесконечных возрастающих цепей.

ЗАМЕЧАНИЕ 1. Нетрудно понять, что в шкале $< W, \leq >$ из класса C для любого непустого подмножества $A \subseteq W$ выполняется хотя бы одно из утверждений:

1) для любого элемента $x \in A$ существует $y \in A$ такой, что $x \leq y$ и y максимален в A,

2) A - конфинальное в $< W, \leq >$.

Для некоторых подмножеств A могут выполняться оба утверждения, например, когда A содержит наибольший элемент множества $< W, \leq >$.

ТЕОРЕМА 1. Для любой позитивной П-схемы $P = \varphi(P, Q_1, \ldots, Q_n)$ существует конечное множество П-формул такое, что в любой модели Крипке из класса CM наименьшее решение определяется формулой из этого множества.

ДОКАЗАТЕЛЬСТВО. Фиксируем частично-упорядоченную модель Крипке $< W, \leq, v >$ из класса CM. В силу предложения 1 можем считать, что все вхождения p в φ модализованы. Сделаем с φ то же преобразование, что и в доказательстве теоремы 1[1]. Пронесем все отрицания к переменным. Вынесем все $\wedge$ наружу. Тогда φ представляется в виде $\bigwedge_s \theta_1^s$, где θ_1^s имеет вид $\psi_1^s \vee \bigvee_i \Lambda \theta_i^s$. При этом формула ψ_1^s не содержит p, а каждая формула θ_i^s имеет вид аналогичный виду формулы θ_1^s, за исключением того, что θ_i^s может содержать p в виде дизъюнктивного слагаемого.

Пусть значением p будет наименьшее решение P схемы $P = \varphi(P, Q_1, \ldots, Q_n)$ на фиксированной модели $< W, \leq, v >$. Для каждой подформулы $\Lambda \theta_i^s$ сделаем следующее: если $\Lambda \theta_i^s$ ложна при этом означивании на всех элементах модели, то заменим подформулу $\Lambda \theta_i^s$ формулы φ на $\perp$. Обозначим полученную из φ после всех замен формулу через γ. Через η_1^s обозначим формулу, полученную этими заменами из θ_1^s. Таким образом γ имеет вид $\bigwedge_s \eta_1^s$.

ЛЕММА 1. Наименьшие решения схем $P = \varphi(P, Q_1, \ldots, Q_n)$ и $P = \gamma(P, Q_1, \ldots, Q_n)$ в модели $< W, \leq, v >$ совпадают.

ДОКАЗАТЕЛЬСТВО.

Если P - наименьшее решение схемы $P = \varphi(P, Q_1, \ldots, Q_n)$, то P - решение схемы $P = \gamma(P, Q_1, \ldots, Q_n)$, т.к. значения формул при замене не изменились. Поэтому, если через P' обозначить наименьшее решение схемы $P' = \gamma(P', Q_1, \ldots, Q_n)$, то $P' \subseteq P$. Если формула $\Lambda \theta_i^s$ была ложна во всей модели при означивании p множеством P, то она останется ложной во всей модели при означивании p множеством P'. Это следует из $P' \subseteq P$ и того, что p входит в $\Lambda \theta_i^s$ только позитивно. Поэтому, в равенстве $P' = \gamma(P', Q_1, \ldots, Q_n)$ можно провести обратные замены и получить $P' = \varphi(P', Q_1, \ldots, Q_n)$. Получаем, что $P \subseteq P'$. Лемма доказана.

ЛЕММА 2. Существует П-формула определяющая наименьшее решение схемы $P = \gamma(P, Q_1, \ldots, Q_n)$ в модели $< W, \leq, v >$.

ДОКАЗАТЕЛЬСТВО. Почти повторяет доказательство теоремы 1[1]. Для каждой η_1^s выполняется один из двух случаев.

1) Все подформулы $\Lambda \theta_i^s$ формулы θ_1^s заменены на $\perp$. Тогда формула η_1^s имеет вид ψ_1^s.

2) В формуле η_1^s осталась хотя бы одна подформула $\Lambda \eta_i^s$. Рассмотрим множество A^s элементов, на которых ложна η_1^s. Допустим, что A^s непусто. Докажем, что A^s не конфинальное. Допустим, что для любого элемента $x \in W$ существует y такой, что $x \leq y$ и $y \nVdash \eta_1^s$. Тогда $y \nVdash \Lambda \eta_i^s$ и, поэтому, $x \nVdash \Lambda \eta_i^s$. Так как при замене значения формул не изменились, то $x \nVdash \Lambda \theta_i^s$. Значит, формула $\Lambda \theta_i^s$ всюду ложна и должна была быть заменена. Противоречие. Поэтому для A^s выполнено первое утверждение замечания 1, т.е. для любого элемента $x \nVdash \eta_1^s$ существует $y \nVdash \eta_1^s$ такой, что $x \leq y$ и y максимален среди элементов, на которых ложна η_1^s.

Итак, если для формулы η_1^s выполняется второй случай, то выше любого элемента, на котором η_1^s опровергается можно выбрать максимальный среди тех, на которых ложна η_1^s. Поэтому для γ проходит доказательство теоремы 1[1]. Единственное отличие состоит в том, что формула γ содержит конъюнктивные слагаемые вида ψ_1^s. Множество элементов, на которых ложна ψ_1^s, может и не иметь максимальных. Тем не менее, конструкция из теоремы 1[1] работает и здесь. При вложении опровергающего дерева для ψ_1^s не нужно искать максимальный элемент, на котором опровергается ψ_1^s. Максимальность была нужна только для того, чтобы на образах ответвляющих элементов опровергалось другое коньюнктивное слагаемое. В данном случае ответвляющих элементов нет.

С учетом этих изменений можно повторить доказательство теоремы 1[1]. Лемма доказана.

Для завершения доказательства теоремы достаточно заметить, что рассматривая все возможные модели $< W, \leq, v >$ из класса C, мы получим лишь конечное множество формул γ. Поэтому, определяющих формул имеется конечное число. Теорема доказана.

Доказательство дает алгоритм построения этого конечного множества определяющих формул. Для этого нужно перебрать все возможные варианты замен формул вида $\wedge\, \theta_i^l$ на $\perp$ и для каждого варианта построить определяющую формулу по алгоритму теоремы 1[1].

Кроме того, заметим, что эта теорема не является обобщением теоремы 4[1], т.к. в теореме [1] достаточно одной формулы.

Следующий пример иллюстрирует конструкцию и показывает, что вообще говоря нельзя обойтись одной П-формулой.

ПРИМЕР 1. Рассмотрим схему $P = \wedge(P \vee Q) \vee \wedge(P \vee \neg Q)$, здесь $\varphi = \wedge(p \vee q) \vee \wedge(p \vee \neg q)$. Возьмем две модели с из класса CM. Каждая из них состоит из множества Z целых чисел с естественным порядком. Значением переменной q в первой модели является множество $Q = \{n \in Z | n - \text{четное или } n \geq 0\}$, во второй - $Q = \{n \in Z | n - \text{четное}\}$. Нетрудно проверить, что наименьшим решением в первой модели будет все множество Z, во второй - пустое множество.

Применим наш алгоритм. Рассмотрим вторую модель. В качестве значения p берем ?. Формулы $\wedge(p \vee q)$ и $\wedge(p \vee \neg q)$ на модели всюду ложны, поэтому мы заменяем их на $\perp$. В этом случае $\gamma = \perp \vee \perp$. Наименьшее решение определяется формулой $\perp$. Рассмотрим первую модель. В качестве значения p берем Z. На первой модели формулы $\wedge(p \vee q)$ и $\wedge(p \vee \neg q)$ не являются всюду ложными, поэтому замен нет. Значит, в этом случае $\gamma = \varphi$. Определяющая формула строится по алгоритму из [1] и эквивалентна $\top$.

Можно доказать, что наименьшее решение во всех моделях из класса CM определяется формулами из множества $\{\perp, \top\}$. Если бы мы перебрали все замены на $\perp$, то нам нужно было бы рассмотреть варианты $\perp \vee \wedge(p \vee \neg q)$ и $\wedge(p \vee q) \vee \perp$. Алгоритм построения определяющих формул дал бы нам в этих случаях формулы эквивалентные $\wedge\neg q$ и $\wedge q$. Т.е. мы получили бы множество $\{\perp, \top, \wedge\neg q, \wedge q\}$.

Не существует П-формулы, которая на первой модели принимает значение Z, а на второй - значение $\varnothing$. Для доказательства этого можно проверить по индукции, что для любой П-формулы α от переменной q существует $n \in Z$ такое, что в первой модели значение формулы α на всех $i \leq n$ совпадает со значением q, $\neg q$, $\perp$ или $\top$, и при этом если α истинна на i в первой модели, то она истинна на i и во второй модели.

Если не ограничивать себя П-формулами, то в качестве определяющей формулы можно взять модальную формулу $\Sigma(\wedge q \vee \wedge\neg q)$. Как будет показано в дальнейших работах в теоремах 1 и 2 вместо конечного числа П-формул можно взять одну модальную формулу. Пример закончен.

Рассмотрим иррефлексивный случай.

Отношение R назовем строгим частичным порядком, если оно иррефлексивно и транзитивно. На любом строго частично-упорядоченном множестве можно ввести частичный порядок по правилу: $x \leq y$ тогда и только тогда, когда $x < y$ или $x = y$.

Зададим класс SC строго частично-упорядоченных шкал следующим определением.

Шкала $< W, < >$ лежит в классе SC, если соответствующая ему частично-упорядоченная шкала $< W, \leq >$ лежит в классе C.

Класс SCM моделей Крипке состоит из моделей $< W, <, v >$ таких, что шкала $< W, < >$ лежит в SC.

Следующий пример показывает нарушение единственности и существования решений модализованных схем в моделях из SCM.

ПРИМЕР 2. Рассмотрим П-схемы $P = \wedge P$, $P \doteq \wedge\neg P$ и модель, состоящую из натуральных чисел N с естественным строгим порядком $0 < 1 < 2 < \ldots$. Эта модель лежит в SCM. Первая схема имеет два решения: пустое множество и N. Вторая схема не имеет решений.

Поэтому для доказательства следующей теоремы применяем конструкцию из [2].

ТЕОРЕМА 2. Для любой позитивной П-схемы $P = \varphi(P, Q_1, \ldots, Q_n)$ существует конечное множество P-формул такое, что в любой модели Крипке из класса SCM наименьшее решение определяется формулой из этого множества.

ДОКАЗАТЕЛЬСТВО. Следуем плану доказательства теоремы 1. Фиксируем строго частично-упорядоченную модель Крипке $< W, <, v >$ из класса SCM. Считаем, что все вхождения p в φ модализованы. Преобразуем φ к виду $\bigwedge_i \theta_1^s$.

Для получения формулы γ делаем ту же замену. Пусть значением p будет наименьшее решение схемы. Для каждой подформулы $\wedge\, \theta_i^s$ сделаем следующее: если $\wedge\, \theta_i^s$ ложна при этом означивании на всех элементах модели, то заменим подформулу $\wedge\, \theta_i^s$ формулы φ на $\perp$. Рассмотрим схему $P = \gamma(P, Q_1, \ldots, Q_n)$. Аналогично доказываются следующие леммы.

ЛЕММА 3. Наименьшие решения схем $P = \varphi(P, Q_1, \ldots, Q_n)$ и $P = \gamma(P, Q_1, \ldots, Q_n)$ в модели $< W, <, v >$ совпадают.

ЛЕММА 4. Существует П-формула определяющая наименьшее решение схемы $P = \gamma(P, Q_1, \ldots, Q_n)$ в модели $< W, <, v >$.

В доказательстве леммы 4 используем конструкцию из [2]. Кроме того, нужен аналог леммы 2[2].

ЛЕММА 5. Если каждая формула η_1^s содержит p, то наименьшее решение определяется формулой $\top$.

ДОКАЗАТЕЛЬСТВО. Через A^s будем обозначать множество элементов, на которых ложна η_1^s . Если каждая формула η_1^s содержит p, то это означает, что в η_1^s осталась хотя бы одна подформула $\wedge\, \eta_i^s$. Отсюда так же как в лемме 2 следует, что для любого $x \in A$ существует максимальный в A^s элемент $x < y$.

Пусть значением p будет наименьшее решение P. Допустим, что P не содержит некоторый элемент x. Тогда на x опровергается γ и, значит, некоторая формула η_1^s . Возьмем элемент $y > x$ максимальный среди элементов, на которых опровергается η_1^s. Так как η_1^s содержит p в $\wedge\, \eta_i^s$, то найдется элемент z такой, что $z > y$ и $z \models p$. Так как $z \not\models \eta_1^s$, то на z опровергается некоторая другая формула η_1^t, $t \neq s$. Поступаем с z так же как и с элементом x. Через конечное число шагов формулы η_1^s закончатся.

Противоречие. Лемма доказана.

Таким образом мы получим конечное множество формул γ и определяющих формул. Теорема доказана.

ПРИМЕР 3. Иррефлексивный аналог примера 1. Пусть $\varphi = \wedge\,(p \vee q)$, т.е. схема $P = \wedge\,(P \vee Q)$. Рассмотрим две модели, состоящих из множества Z целых чисел с естественным строгим линейным порядком. Значением переменной q в первой модели является множество $Q = \{n \in Z | n \geq 0\}$, во второй $Q = \emptyset$. Наименьшим решением в первой модели будет все множество Z, во второй - пустое множество. Рассмотрим первую модель. В качестве значения p берем Z. Формула $\wedge\,(p \vee q)$ не является всюду ложной, поэтому ее не заменяем. По лемме 5 наименьшее решение определяется формулой $\top$. Рассмотрим вторую модель. В качестве значения p берем $\emptyset$. Формула $\wedge\,(p \vee q)$ всюду ложна, заменяем ее на $\perp$. Получаем $\gamma = \perp$. Наименьшее решение определяется формулой $\perp$.

Так как имеются только две возможности: формула $\varphi = \wedge\,(p \vee q)$ заменяется и не заменяется, то в любой модели Крипке из класса SCM наименьшее решение определяется формулой из множества $\{\perp, \top\}$. Можно доказать, что не существует единой определяющей П-формулы. Если не ограничивать себя П-формулами, то в качестве определяющей формулы можно взять формулу $\boxminus \wedge q$. Пример закончен.

Итак, при обрыве возрастающих цепей достаточно одной П-формулы, а при условии конфинальности бесконечных возрастающих цепей нужно конечное множество. Рассмотрим схему и по теоремам 1 и 2 построим для нее эти множества в рефлексивном и иррефлексивном случаях. Ясно, что объединив эти множества получим множество, пригодное для обоих случаев. А можно ли в случае обрыва возрастающих цепей построить одну П-формулу, пригодную в рефлексивном и иррефлексивном случаях? Как показывает следующий пример, единого алгоритма не существует.

ПРИМЕР 4. Схема $P = \wedge\, P$. Докажем, что не существует П-формулы без переменных, которая определяет наименьшее решение этой П-схемы в рефлексивном и иррефлексивном случаях одновременно. Рассмотрим две модели, состоящих из множества N^- неположительных целых чисел с естественным порядком. На первой задан строгий линейный порядок

$\ldots < -2 < -1 < 0$, на второй - линейный порядок $\ldots \leq -2 \leq -1 \leq 0$.
Обе являются моделями с обрывом возрастающих цепей.

Наименьшим решением в первой модели будет все множество N^-, на второй - пустое множество. Докажем, что не существует П-формулы без переменных, которая на первой модели принимает значение N^-, а на второй - значение ϕ.

Ясно, что любая П-формула принимает на второй модели значение ϕ или N^-.

Индукцией докажем, что для любой П-формулы α без переменных существует $n \in N^-$ такое, что формула α в первой модели либо ложна на всех $i \leq n$, либо истинна, и при этом если α истинна на i в первой модели, то она истинна на i и во второй модели.

БАЗИС. Если α не содержит модальностей, то на обеих моделях одновременно она принимает значения ? или N^-.

ШАГ. Случаи $\wedge$ и $\vee$ тривиальны.

Случай $\alpha = \wedge\beta$. Пусть, например, в первой модели β ложна при $i \leq n$. Тогда $\wedge\beta$ ложна на всех $i \leq n-1$. Утверждение выполняется. Если в первой модели β истинна на всех $i \leq n$, то из индукционного предположения следует, что на второй модели она всюду истинна и поэтому на второй модели $\wedge\beta$ всюду истинна. На первой же модели $\wedge\beta$ либо ложна на всех $i \leq n$, либо истинна. Пример закончен.

Теоремы 1 и 2 имеют следствия для логик. Через L_1 обозначаем модальную пропозициональную логику, характеризуемую всеми шкалами Крипке из класса C, а через L_2 - логику, характеризуемую всеми шкалами из класса SC (о понятии характеризации можно прочитать например в [4]).

СЛЕДСТВИЕ 1. Для любой позитивной П-схемы $P = \varphi(P, Q_1, \ldots, Q_n)$ существует конечное множество П-формул $\omega_i(q_1, \ldots, q_n)$ такое, что выполняется

$$\vdash_{L_1} \bigvee_i (\wedge(\omega_i \equiv \varphi(\omega_i, q_1, \ldots, q_n)) \wedge (\wedge(p \equiv \varphi(p, q_1, \ldots, q_n)) \Rightarrow (\omega_i \Rightarrow p)))$$

СЛЕДСТВИЕ 2. Для любой позитивной П-схемы $P = \varphi(P, Q_1, \ldots, Q_n)$ существует конечное множество П-формул $\omega_i(q_1, \ldots, q_n)$ такое, что выполняется

$$\vdash_{L_2} \bigvee_i (\Gamma(\omega_i \equiv \varphi(\omega_i, q_1, \ldots, q_n)) \wedge (\Gamma(p \equiv \varphi(p, q_1, \ldots, q_n)) \Rightarrow (\omega_i \Rightarrow p)))$$

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О ПРОСТЫХ СКЕЛЕТАХ КОНГРУЭНЦ-МОДУЛЯРНЫХ МНОГООБРАЗИЙ[1]

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В серии работ А.Г. Пинуса, включающей обзорную статью [1] и мо-нографию [2], были введены и детально изучены для случая конгруэн-дистрибутивных многообразий понятия скелетов многообразий: скеле-эпиморфности и скелета вложимости. В работе [3] автора часть резул-татов обобщается на конгруэнц-модулярные многообразия, содержащ-подпрямо неразложимую алгебру с неабелевым монолитом. В настоящ-работе излагаются новые факты полученные автором в данном направле-нии, а также некоторые интересные результаты относительно абелевы-конгруэнц-модулярных многообразий.

Напомним основные определения и обозначения. Если $\mathfrak{K}$ — некоторы-класс алгебр, то через $\Im\mathfrak{K}$ обозначим совокупность типов изоморфизм-$\mathfrak{K}$-алгебр. Если $a, b \in \Im\mathfrak{K}$, то $a \leq b$ $(a \ll b)$ имеет место тогда и тольк-тогда, когда алгебра с типом изоморфизма a является подалгеброй (го-моморфным образом) алгебры с типом изоморфизма b. Отношения $\leq, \ll$ на $\Im\mathfrak{K}$ являются отношениями квазипорядка. Для кардинала $\aleph$ через $\mathfrak{K}_\aleph$ обозначим совокупность $\mathfrak{K}$-алгебр мощности не большей чем $\aleph$. Скеле-том вложимости многообразия $\mathcal{M}$ (скелетом эпиморфности $\mathcal{M}$) называет-ся квазиупорядоченный класс $\langle \Im\mathcal{M}; \leq \rangle$ $(\langle \Im\mathcal{M}; \ll \rangle)$; $\aleph$-ограниченным ске-летом вложимости (эпиморфности) называется $\langle \Im\mathcal{M}_\aleph; \leq \rangle$ $(\langle \Im\mathcal{M}_\aleph; \ll \rangle)$. Счетным скелетом эпиморфности (вложимости) многообразия $\mathcal{M}$ назо-вем квазиупорядоченное множество $\langle \Im\mathcal{M}_{\aleph_0}; \ll \rangle$ $(\langle \Im\mathcal{M}_{\aleph_0}; \leq \rangle)$.

Для квазиупорядоченного множества $\langle A; \leq \rangle$ через $\equiv_\leq$ обозначим экви-валентность на A, естественным образом связанную с квазипорядком $\leq$

[1]Работа выполнена при финансовой поддержке Российского фонда фундаментальных исследований (код проекта 96-01-01675)

$(a \equiv_\leq b$ тогда и только тогда, когда $a \leq b$ и $b \leq a)$. Через $\langle A/\equiv_\leq; \leq \rangle$ будем обозначать результат факторизации квазипорядка $\langle A; \leq \rangle$ по $\equiv_\leq$, и квазиупорядоченное множество $\langle A; \leq \rangle$ назовем фактор-линейно упорядоченным, если множество $\langle A/\equiv_\leq; \leq \rangle$ линейно упорядочено.

Через $Con\mathcal{A}$ $(Con_p\mathcal{A})$ будем обозначать решетку конгруэнций алгебры $\mathcal{A}$ (частично упорядоченное множество главных конгруэнций алгебры $\mathcal{A}$); $\theta^{\mathcal{A}}(a,b)$ — главная конгруэнция на алгебре $\mathcal{A}$, порожденная парой ее элементов $\langle a,b \rangle$; $\mathcal{M}(\mathcal{A})$ — многообразие, порожденное алгеброй $\mathcal{A}$. Если $\mathcal{M}$ — произвольное многообразие, $\aleph$ — некоторый кардинал, то $\mathcal{F}_{\mathcal{M}}(\aleph)$ — свободная $\aleph$-порожденная алгебра многообразия $\mathcal{M}$; $\mathcal{M}'$ — совокупность одноэлементных $\mathcal{M}$-алгебр.

Для любой алгебры $\mathcal{A}$ и булевой алгебры $\mathcal{B}$ через $\mathcal{A}^{\mathcal{B}}$ обозначим булеву степень алгебры $\mathcal{A}$ по булевой алгебре $\mathcal{B}$ (определение см., напр., в [1]), через $\mathcal{B}^*$ — стоуновское пространство булевой алгебры $\mathcal{B}$. Если $\mathcal{A}_1$ — подалгебра декартовой степени $\mathcal{A}^I$ алгебры $\mathcal{A}$, то для $i \in I$ символом π_i обозначаем i-е проектирование $\mathcal{A}_1$, т. е. гомоморфизм $\mathcal{A}_1$ в $\mathcal{A}$ такой, что для $a \in A_1$ $\pi_i(a) = a(i)$. Пусть $ker\ \pi_i$ — ядро гомоморфизма π_i. Если $a, b \in A_1$, то $[[a = b]] = \{i \in I | b(i) = a(i)\}$, $[[a \neq b]] = I \setminus [[a = b]]$. В дальнейшем рассматриваются лишь алгебры не более чем счетной сигнатуры, все многообразия предполагаются конгруэнц-модулярными.

Отметим следующие простые факты, связанные со счетными скелетами эпиморфности произвольных многообразий. Квазиупорядоченное множество $\langle \Im\mathcal{M}_{\aleph_0}; \ll \rangle$ содержит наименьший элемент (тип изоморфизма одноэлементной алгебры), хотя бы один элемент, покрывающий этот наименьший (тип изоморфизма простой $\mathcal{M}$-алгебры, существующей по теореме Магари [4]). На самом деле типы изоморфизма всех простых $\mathcal{M}$-алгебр суть покрывающие наименьший элемент в $\langle \Im\mathcal{M}; \ll \rangle$. Заметим также, что классы $\equiv_\ll$-эквивалентности, содержащие эти покрывающие, одноэлементны. Кроме того, $\langle \Im\mathcal{M}_{\aleph_0}; \ll \rangle$ содержит наибольший элемент (тип изоморфизма алгебры $\mathcal{F}_{\mathcal{M}}(\aleph_0)$).

Рассмотрим конгруэнц-модулярное многообразие $\mathcal{M}$, счетный скелет эпиморфности которого фактор-линейно упорядочен. Такое многообразие содержит в точности одну простую алгебру (не более чем счетную), которую мы обозначим через $\mathcal{A}$. Сначала рассмотрим случай, когда $\mathcal{A}$ неабелева.

ТЕОРЕМА 1. Если $\mathcal{M}$ — конгруэнц-модулярное многообразие и $\mathcal{A}$ — неабелева простая $\mathcal{M}$-алгебра, то для любых булевых алгебр $\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2$:

а) для $f,g,h,k \in \mathcal{A}^{\mathcal{B}}$ $\langle f,g\rangle \in \theta^{\mathcal{A}^{\mathcal{B}}}(h,k)$ тогда и только тогда, когда $[[h=k]] \subseteq [[f=g]]$,

б) $Con_p(\mathcal{A}^{\mathcal{B}}) \cong \mathcal{B}$,

в) для любой $\theta \in Con(\mathcal{A}^{\mathcal{B}})$ существует $\psi \in Con\mathcal{B}$ такая, что $\mathcal{A}^{\mathcal{B}}/\theta \cong \mathcal{A}^{\mathcal{B}/\psi}$,

г) отношения $\mathcal{A}^{\mathcal{B}_1} \ll \mathcal{A}^{\mathcal{B}_2}$ и $\mathcal{B}_1 \ll \mathcal{B}_2$ эквивалентны,

д) если многообразие $\mathcal{M}$ имеет продолжимые конгруэнции, то отношения $\mathcal{A}^{\mathcal{B}_1} \leq \mathcal{A}^{\mathcal{B}_2}$ и $\mathcal{B}_1 \leq \mathcal{B}_2$ эквивалентны для любых неодноэлементных $\mathcal{B}_1$ и $\mathcal{B}_2$.

Утверждения а) и б) данной теоремы являются частным случаем теоремы, доказанной в работе [3] автора. Утверждения в), г) и д) являются аналогами соответствующих утверждений для алгебр конгруэнц-дистрибутивных многообразий, которые рассматривались в статье [5], с учетом этого их подробное доказательство не представлят труда. (См. также работу [3] автора.)

В силу определения булевой степени, если булева алгебра $\mathcal{B}$ не является конечнопорожденной, то алгебра $\mathcal{A}^{\mathcal{B}}$ также не может быть конечнопорожденной. Пусть $\mathcal{B}_F$ — булева алгебра конечных и ко-конечных подмножеств счетного множества. Для любого конечного n в силу фактор-линейной упорядоченности счетного скелета эпиморфности $\mathcal{M}$ либо $\mathcal{F}_{\mathcal{M}}(n) \ll \mathcal{A}^{\mathcal{B}_F}$, либо $\mathcal{A}^{\mathcal{B}_F} \ll \mathcal{F}_{\mathcal{M}}(n)$. Но $\mathcal{B}_F$, а значит, и $\mathcal{A}^{\mathcal{B}_F}$ не являются конечнопорожденными. В силу этого имеет место $\mathcal{F}_{\mathcal{M}}(n) \ll \mathcal{A}^{\mathcal{B}_F}$ и $\mathcal{A}^{\mathcal{B}_F} \not\ll \mathcal{F}_{\mathcal{M}}(n)$. Алгебра $\mathcal{F}_{\mathcal{M}}(n)$ должна иметь вид $\mathcal{A}^{\mathcal{B}}$, где $\mathcal{B}$ — некоторый гомоморфный образ $\mathcal{B}_F$. Все неизоморфные $\mathcal{B}_F$ ее гомоморфные образы суть конечные булевы алгебры. Таким образом, для любого $n \in \omega$ $\mathcal{F}_{\mathcal{M}}(n) \in \mathcal{M}(\mathcal{A})$ и существует $m \in \omega$ такое, что $\mathcal{F}_{\mathcal{M}}(3) \cong \mathcal{A}^m$. Конгруэнции алгебры $\mathcal{A}^m$ образуют дистрибутивную решетку; по теореме Йонссона многообразие $\mathcal{M}$ конгруэнц-дистрибутивно. А конгруэнц-

дистрибутивные многообразия с простыми счетными скелетами эпиморфности уже описаны в статье [6]. Сформулируем полученный результат.

ТЕОРЕМА 2. Счетный скелет эпиморфности неабелевого конгруэнц-модулярного многообразия фактор-линейно упорядочен тогда и только тогда, когда это многообразие порождается квазипримальной алгеброй без собственных подалгебр. Из фактор-линейной упорядоченности счетного скелета эпиморфности неабелевого конгруэнц-модулярного многообразия $\mathcal{M}$ следует совпадение счетных скелетов эпиморфности и вложимости $\mathcal{M}$.

Пусть теперь алгебра $\mathcal{A}$ абелева. Рассмотрим булеву степень $\mathcal{A}^{\mathcal{B}_F}$. Она не является конечнопорожденной алгеброй, т. е. $\mathcal{A}^{\mathcal{B}_F} \not\ll \mathcal{F}_{\mathcal{M}}(n)$ для любого $n \in \omega$. Из условия фактор-линейной упорядоченности $\langle \Im\mathcal{M}_{\aleph_0}; \ll\rangle$ получаем: для всякого $n \in \omega$ имеет место $\mathcal{F}_{\mathcal{M}}(n) \ll \mathcal{A}^{\mathcal{B}_F}$. Как хорошо известно, это влечет совпадение $\mathcal{M}$ и $\mathcal{M}(\mathcal{A})$, многообразие $\mathcal{M}$ порождается алгеброй $\mathcal{A}$ и является абелевым.

По теореме Фриза-Маккензи [7] многообразие $\mathcal{M}$ полиномиально эквивалентно многообразию унитарных R-модулей над некоторым кольцом R с единицей. Предположим, что алгебра $\mathcal{A}$ содержит собственную неодноэлементную подалгебру $\mathcal{A}_1$. Возьмем произвольный элемент $a \in \mathcal{A}_1$ и по паре $\langle \mathcal{A}, a\rangle$ построим R-модуль $M_{\mathcal{A}}$. Этот модуль, с одной стороны, должен быть простым, а с другой, он должен содержать собственный неодноэлементный подмодуль, соответствующий алгебре $\mathcal{A}_1$. Полученное противоречие показывает, что алгебра $\mathcal{A}$ не содержит собственных неодноэлементных подалгебр.

Вернемся к рассмотрению алгебры $\mathcal{A}^{\mathcal{B}_F}$. Зафиксируем два произвольных элемента $a, b \in \mathcal{A}$, по доказанному они порождают алгебру $\mathcal{A}$. На алгебре $\mathcal{A}^{\mathcal{B}_F}$ будем задавать структуру модуля. Пусть a^0, b^0 — такие элементы алгебры $\mathcal{A}^{\mathcal{B}_F}$, что $a^0(i) = a$, $b^0(i) = b$ для всех $i \in \omega$. Для каждого $j \in \omega$ определим элемент $c^j \in \mathcal{A}^{\mathcal{B}_F}$ так, что $c^j(i) = a$ при $i \neq j$ и $c^j(j) = b$. По паре $\langle \mathcal{A}^{\mathcal{B}_F}, a^0\rangle$ строим R-модуль M_ω. Каждая пара $\langle a^0, c^j\rangle$, а также пара $\langle a^0, b^0\rangle$, порождают в алгебре $\mathcal{A}^{\mathcal{B}_F}$ подалгебры, изоморфные алгебре $\mathcal{A}$. Этим подалгебрам соответствуют простые подмоду M_j и N в модуле M_ω (каждый из которых изоморфен модулю $M_{\mathcal{A}}$). Чи тателям предоставляется проверить, что модуль M_ω является прям суммой указанных подмодулей, т. е. он вполне приводим. Что еще да нам условие $\mathcal{F}_{\mathcal{M}}(n) \ll \mathcal{A}^{B_r}$? По заданному $\mathcal{M}$-гомоморфизму мы всег можем построить соответствующий гомоморфизм R-модулей, значит, ка ждой свободной алгебре $\mathcal{F}_{\mathcal{M}}(n)$ соответствует некоторый фактор-моду модуля M_ω, а они нам известны, это конечные прямые суммы просты R-модулей типа $M_{\mathcal{A}}$. Итак, каждая свободная алгебра $\mathcal{F}_{\mathcal{M}}(n)$ конечно ранга является конечной прямой степенью алгебры $\mathcal{A}$.

Далее мы докажем, что каждой $\mathcal{M}$-алгебре соответствует вполне пр водимый R-модуль. Пусть $\mathcal{A}' \in \mathcal{M}$ и $c, d \in \mathcal{A}'$. Рассмотрим подалгеб $\mathcal{A}_d \leq \mathcal{A}'$, порожденную элементами c, d. В качестве порождающих эл ментов алгебры $\mathcal{F}_{\mathcal{M}}(2)$ возьмем u и v. По парам $(\mathcal{A}', c)$, $(\mathcal{A}_d, c)$, $(\mathcal{F}_{\mathcal{M}}(2), $ строим соответствующие R-модули $M(\mathcal{A}', c)$, $M(\mathcal{A}_d, c)$, $M(\mathcal{F}_{\mathcal{M}}(2),$ Модуль $M(\mathcal{A}_d, c)$ является гомоморфным образом модуля $M(\mathcal{F}_{\mathcal{M}}(2),$ и, следовательно, разлагается в сумму простых R-модулей. В силу пр извольности элемента $d \in \mathcal{A}'$ модуль $M(\mathcal{A}', c)$ также является суммой (н прямой) простых R-модулей. По соответствующей теореме теории ко модуль $M(\mathcal{A}', c)$ вполне приводим. В рассматриваемом нами многообр зии R-модулей все простые модули изоморфны, поэтому тип изоморф ма определяется числом слагаемых в прямой сумме. В итоге, линейн оказывается не только счетный скелет эпиморфности $\langle \Im\mathcal{M}_{\aleph_0}; \ll \rangle$, но в скелеты эпиморфности многообразия $\mathcal{M}$.

ЛЕММА. Пусть A — простая абелева алгебра без неодноэлементн собственных подалгебр. Пусть $\{a_1\}, \{a_2\}$ — подалгебры алгебры $\mathcal{A}$ счетный скелет эпиморфности многообразия $\mathcal{M} = \mathcal{M}(\mathcal{A})$ фактор-линей упорядочен. Тогда существует автоморфизм φ алгебры $\mathcal{A}$ такой, ч $\varphi(a_2) = a_1$.

Доказательство. Выберем подалгебры $\mathcal{A}_1$, $\mathcal{A}_2$ алгебры $\mathcal{A}^{B_r}$ так, ч $\mathcal{A}_j = \{f \in \mathcal{A}^{B_r}$ — существует $n \in \omega$ такой, что для всех $i \leq n$ $f(i) = a_j$ Символами $a_1{}^0, a_2{}^0$ обозначаем такие элементы алгебры $\mathcal{A}^{B_r}$, что $a_j{}^0(i)$ a_j для всех $i \in \omega$. В силу фактор-линейной упорядоченности $\langle \Im\mathcal{M}_{\aleph_0};$

жно считать, что $\mathcal{A}_1 \ll \mathcal{A}_2$ и пусть ψ — эпиморфизм $\mathcal{A}_2$ на $\mathcal{A}_1$. Фактор алгебры $\mathcal{A}_j$ по любой главной конгруэнции изоморфен самой $\mathcal{A}_j$. Поэтому если n таково, что для $i \leq n$ $\psi(a_2{}^0)(i) = a_1$, то, факторизуя $\mathcal{A}_1$ по $\theta(a_1{}^0, \psi(a_2{}^0))$, получаем эпиморфизм ψ_1 алгебры $\mathcal{A}_2$ на алгебру $\mathcal{A}_1$ такой, что $\psi(a_2{}^0) = a_1{}^0$. Пусть b — такой элемент алгебры $\mathcal{A}_2$, что $b(0) \neq a_2$ и $b(i) = a_2$ для всех $i > 0$. Если $\mathcal{A}_3$ — подалгебра алгебры $\mathcal{A}_2$, порожденная элементами b и $a_2{}^0$, то $\mathcal{A}_3 \cong \mathcal{A}$. Пусть i таково, что $\psi_1(b)(i) \neq a_1$. Тогда, так как $\psi_1(\mathcal{A}_3) \cong \mathcal{A}$ проста, то $\pi_i\psi_1$ — изоморфизм алгебры $\mathcal{A}_3$ на алгебру $\mathcal{A}$, и при этом $\pi_i\psi_1(a_2{}^0) = a_1$. А так как $\langle \mathcal{A}_3, a_2{}^0 \rangle \cong \langle \mathcal{A}, a_2 \rangle$, то получаем искомый автоморфизм алгебры $\mathcal{A}$, переводящий a_2 в a_1. Лемма доказана.

Теперь мы можем сформулировать полученный результат.

ТЕОРЕМА 3. Пусть счетный скелет эпиморфности абелевого конгруэнц-модулярного многообразия $\mathcal{M}$ фактор-линейно упорядочен. Тогда многообразие $\mathcal{M}$ порождается простой алгеброй $\mathcal{A}$ без неодноэлементных собственных подалгебр, если $\mathcal{A}$ имеет несколько одноэлементных подалгебр, то они переводимы друг в друга автоморфизмами алгебры $\mathcal{A}$, все скелеты эпиморфности многообразия $\mathcal{M}$ фактор-линейно упорядочены.

Пример векторных пространств над полем рациональных чисел показывает, что порождающая простая алгебра может быть бесконечной. Остается открытым

ВОПРОС. Существует ли бесконечная алгебра конечной сигнатуры, порождающая конгруэнц-модулярное многообразие алгебр с фактор-линейным счетным скелетом эпиморфности.

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НЕКОТОРЫЕ ВОПРОСЫ СЛАБО О-МИНИМАЛЬНЫХ БИНАРНЫХ ТЕОРИЙ

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В этой статье мы исследуем некоторые свойства слабо о-минимальных бинарных теорий, о-минимальный вариант которых рассматривался в [1].

Определение 1 *Пусть A подмножество линейно упорядоченного множества B. Тогда A выпукло, если для любых $\alpha, \beta \in A$ следующее выполняется:*

$$\forall \gamma \in B \ [\alpha < \gamma < \beta \Rightarrow \gamma \in A].$$

Определение 2 i. *Линейно упорядоченная модель M сигнатуры Σ слабо о-минимальна, если реализация любой формулы сигнатуры $\Sigma(M)$ от одной переменной - объединение конечного числа непересекающихся выпуклых множеств.*

ii. *Теория T называется слабо о-минимальной, если любая модель этой теории слабо о-минимальна.*

Определение 3 *[1] Теория T называется n-арной, если любая формула этой теории более чем n переменных без параметров эквивалентна булевой комбинации формул от n свободных переменных. 2-арная теория называется бинарной.*

Определение 4 *[2] Пусть $p \in S_1(A), B \subset M$, такое что M $|B \cup A|^+$-насыщенная. Окрестностью множества B в типе p называется следующее множество:*

$$V_p(B) := \{\gamma \in M | \exists \gamma_1, \gamma_2 \in p(M), \exists H(x, \bar{b}, \bar{c}), \bar{b} \in B, \bar{c} \in A,$$
$$\gamma_1 < H(M, \bar{b}, \bar{c}) < \gamma_2, \gamma \in H(M, \bar{b}, \bar{c})\}$$

Определение 5 (Б.С.Байжанов) *Множество $B \subset M$ называется n-независимым над A, если $\forall C \subset B, |C| = n$, $\forall \alpha \in C$, $p_\alpha = tp(\alpha|A), \exists \gamma_1, \gamma_2 \in p_\alpha(M)\ \alpha \in (\gamma_1, \gamma_2)$, $\forall \gamma \in (\gamma_1, \gamma_2) \cup V_{p_\alpha}(\alpha)\ tp(\gamma|A \cup C \setminus \alpha) = tp(\alpha|A \cup C \setminus \alpha)$.*

Множество B называется независимым над A, если оно n-независимо над A для любого конечного n.

Замечание 1 *Пусть T o-минимальная теория, тогда если $B \subset$ $Mod(T)$ независимо над A, то B алгебраически независимо над A.*

Доказательство. Допустим обратное, B алгебраически зависимо над A, т.е. $\exists \alpha \in B\ \alpha \in acl(A \cup B \setminus \alpha) =$

$$= \{\delta | \exists \phi(x, \bar\beta, \bar a),\ \bar\beta \in B \setminus \alpha\ M \models \exists^1 x \phi(x, \bar\beta, \bar a) \& \phi(\delta, \bar\beta, \bar a)\}.$$

Что противоречит определению независимости B. Следовательно, $\forall \gamma \in p_\alpha(M) \setminus \alpha\ tp(\gamma|A \cup B \setminus \alpha) \neq tp(\alpha|A \cup B \setminus \alpha)$.

Определение 6 *Будем говорить, что теория T является тривиальной, если для любой $M \models T$ и любого $B \subset M$ всякий раз, когда множество B является 2-независимым над A, оно независимо над A.*

Предложение 1 *Пусть T слабо o-минимальная теория. Если T бинарна, то T тривиальна.*

Доказательство. Допустим обратное. Существуют модель M теории T и множество $B \subset M$ 2-независимое, но n-зависимое. Поэтому и так как T слабо o-минимальна, существует $\alpha \in B$, существует $\Phi(x, \bar\beta, \bar a)$, где $\bar\beta \in B \setminus \alpha$, $\bar a \in A$, такие что $\forall \gamma_1, \gamma_2 \in p_\alpha(M), \alpha \in (\gamma_1, \gamma_2)$

$$V_{p_\alpha}(\alpha)(\gamma_1, \gamma_2) \cap \Phi(M, \bar\beta, \bar a) \neq \emptyset,$$

$$V_{p_\alpha}(\alpha)(\gamma_1, \gamma_2) \cap \neg\Phi(M, \bar\beta, \bar a) \neq \emptyset,$$

$$\Phi(M, \bar\beta, \bar a) < \neg\Phi(M, \bar\beta, \bar a),\ \alpha \in \neg\Phi(M, \bar\beta, \bar a).$$

Так как T бинарна, то

$$\Phi(x, \bar y, \bar z) \equiv \bigvee_k (\bigwedge_i \phi_j^k(x, y_j) \& \bigwedge \psi_j^k(x, z_j) \& \theta^k(\bar y, \bar z)).$$

Тогда $\Phi(M, \bar\beta, \bar a) = \bigcup_k (\bigcap_j \phi_j^k(M, \beta_j) \cap \bigcap_j \psi_j^k(M, a_j)$.

Выберем в этом объединении множество, определяющее правую границу $\Phi(M, \bar\beta, \bar a)$. Пусть это

$$\bigcap_j \phi_j^k(M, \beta_j) \cap \bigcap_i \psi_i^k(M, a_i)$$

для некоторого фиксированного k. Так как T слабо o-минимальна и $\alpha \notin \Phi(M, \bar\beta, \bar a) (\Rightarrow \forall i, j, k\ \alpha \notin \cap_j \phi_j^k(M, \beta_j) \cap \cap_i \psi_i^k(M, a_i))$, можем выбрать выпуклые подмножества $\phi_j^0(M, \beta_j) \subset \phi_j^k(M, \beta_j)$, $\psi_j^0(M, a_j) \subset \psi_j^k(M, a_j)$, такие что каждое из них содержит правую границу множества, $(\Rightarrow \forall \gamma_1 \in \Phi(M, \bar\beta, \bar a) \cap p_\alpha(M)\ \exists \gamma\ \gamma_1 < \gamma < \neg\Phi(M, \bar\beta, \bar a)\ \gamma \in \phi_j^0(M, \beta_j) \cap \psi^0(M, \bar a))$, и $\exists j\ \alpha \notin \phi_j^0(M, \beta_j) \cap \psi^0(M, \bar a)$. Следовательно, $\forall \gamma_1 \exists \gamma\ \gamma_1 < \gamma < \alpha\ tp(\alpha|\beta_j \cup A) \neq tp(\gamma|\beta_j \cup A)$, т.е. B 2-зависимо над A. Противоречие.

Следствие 1 *Из предложения следует необходимое условие критерия бинарности o-минимальных теорий из статьи Mekler, Rubin, Steinhorn [1]: T бинарная o-минимальная теория тогда и только тогда, когда для любой модели M теории T для любого n для любого $\{a_1, \ldots, a_n\} \subset M$ такое, что если оно алгебраически попарно независимо над A, то оно независимо над A.*

Определение 7 (Б.С.Байжанов) *Теория T квазибинарна, если для любой модели M теории T, для любого $B \subset M$ как только B упорядоченно 2-неразличимо над A, то B упорядоченно n-неразличимо над A для любого натурального n.*

Предложение 2 *Пусть T теория $< M, <, \ldots >$, где $<$ - линейный порядок на M. Если T бинарна, тогда T квазибинарна.*

Доказательство. Пусть существует $M \models T$, $B \subset M$, такие что $\exists n\ B$ упорядоченно n-различимо над A, т.е.

$$\exists \bar\phi(\bar x, \bar a),\ l(\bar x) = n,\ \bar a \in A,\ \exists c_1 < \ldots < c_n, d_1 < \ldots < d_n \in B$$

$$M \models \phi(\bar c, \bar a) \& \neg\phi(\bar d, \bar a).$$

Так как T бинарная, то

$$\phi(\bar{x},\bar{y}) \equiv \bigvee_k(\bigwedge_{i<j} \phi^k_{ij}(x_i,x_j)\& \bigwedge_l \psi^k_l(x_i,\bar{y})\&\theta^k(\bar{y})).$$

$$M \models \bigvee_k(\bigwedge_{i<j} \phi^k_{ij}(c_i,c_j)\& \bigwedge_l \psi^k_l(c_l,\bar{a})\&\theta^k(\bar{a}))$$

$$M \models \bigwedge_k(\bigvee_{i<j} \neg\phi^k_{ij}(d_i,d_j) \vee \bigvee_l \neg\psi^k_l(d_l,\bar{a}) \vee \neg\theta^k(\bar{a}))$$

Т.е. существуют i,j,k такие, что

$$M \models \phi^k_{ij}(c_i,c_j)\&\neg\phi^k_{ij}(d_i,d_j)$$

Отсюда следует, что B упорядоченно 2-различимо над A. Противоречие.

Определение 8 *[2] 2 – A-формула $F(x,y,\bar{a})$ $(G(x,y,\bar{a}))$, $\bar{a}\in A$ называется выпуклой вправо (влево), если*

$$M \models \forall x\forall y[F(x,y,\bar{a}) \to (y \le x \wedge \forall z(y \le z \le x \to F(z,y,\bar{a}))]$$

$$(M \models \forall x\forall y[G(x,y,\bar{a}) \to (x \le y \wedge \forall z(x \le z \le y \to G(z,y,\bar{a}))]).$$

Определение 9 *[2] Пусть $p\in S_1(A)$. 2 – A-формула $\Phi(x,y,\bar{a})$, $\bar{a}\in A$ называется p-устойчивой, если*

$$\forall\alpha \models p\ \exists\gamma_1,\gamma_2 \models p\ \gamma_1 < \Phi(M,\alpha,\bar{a}) < \gamma_2.$$

Определение 10 *[2] Пусть $p,q\in S_1(A)$, p почти ортогонально q ($p\perp^a q$), если*

$$\exists\alpha\in p(M)\ (\equiv \forall\alpha\in p(M))\ V_p(\alpha)=\emptyset.$$

Предложение 3 *Пусть T слабо о-минимальная бинарная теория, $M\models T$, $A\subset M$, $p,q\in S_1(A)$.*

i. Для любой выпуклой вправо или влево p-стабильной 2-A-формулы $F(x,y,\bar{a})$ существует 2–$\emptyset$-определимая формула $\Phi(x,y)$, такая что $\forall\alpha\models p$ $F(M',\alpha,\bar{a})\equiv\Phi(M,\alpha)$.

ii. $p\not\perp^a q$. Тогда $\forall\alpha\models p$, для любой $\Phi(x,\alpha,\bar{a})$, $\Phi(M,\alpha,\bar{a})\subset V_q(\alpha)$ существует 2 – $\emptyset$-определимая формула $\Psi(x,y)$, такая что

$$\Psi(M',\alpha)\equiv\Phi(M,\alpha,\bar{a}).$$

Доказательство. i) Так как T бинарна, то

$$F(x,y,\bar{z}) \equiv \bigvee_k(\phi^k(x,y)\& \bigwedge_j \psi^k_j(x,z_j)\& \bigwedge_j \tau^k_j(y,z_j)),$$

тогда

$$\forall\alpha\models p\ F(M,\alpha,\bar{a}) = \bigcup_k(\phi^k(M,\alpha)\cap\bigcap_j \psi^k(M,a_j)),$$

так как $p(M)\subset\psi^k_j(M,a_j)$, а $F(M,\alpha,\bar{a})\subset p(M)$, то $F(M,\alpha,\bar{a})\equiv$ $\bigcup_k\psi^k(M,\alpha)$.

Доказательство ii) аналогично.

Предложение 4 *Пусть T бинарная слабо о-минимальная теория, M модель T. Тогда для любой одноместной A-определимой функции $y=f(x,\bar{d})$, $\bar{d}\in A$ $\exists k<\omega$ $\exists\phi_1(x,\bar{d}),\phi_2(x,\bar{d}),\ldots,phi_k(x,\bar{d})$, такие что множества, которые они определяют выпуклые и $\forall i$ $1\le i\le k$ $\exists g_i(x)$ или $\exists c_i$ $\forall x\in\phi_i(M,d)$ $f(x,\bar{d})=g_i(x)$ или $f(x,\bar{d})=c_i$.*

Доказательство. Так как T бинарна, то

$$M\models y=f(x,\bar{d}) \leftrightarrow \bigvee_j(\psi^j(x,y)\&\theta^j(x,\bar{d})\&\tau^j(y,\bar{d}))$$

Так как предложение

$$M\models\forall x\ \exists^1 y(y=f(x,\bar{d})),$$

то

$$M\models\bigvee_j[\exists y^1(\psi^j(x,y)\&\theta^j(x,\bar{d})\&\tau^j(y,\bar{d}))].$$

Значит для любого j хотя бы одна из формул $\psi^j(x,y)$ и $\tau^j(y,\bar{d})$ эквивалентна $\emptyset$-определимым или константным функциям соответственно и на каждом $\theta^j(M,\bar{d})$ $y=f(x,\bar{d})$ эквивалентна конечному числу константных и $\emptyset$-определимых функций, потому что T слабо о-минимальна и для любых j множества

$$\{x|\exists^1 y(y=f(x,\bar{d})\&\theta^j(x,bard)\&y=f(x,\bar{d})\leftrightarrow\psi^j(x,y))\},$$

$$\{x|\exists^1 y(y=f(x,\bar{d})\&\theta^j(x,bard)\&y=f(x,\bar{d})\leftrightarrow\tau^j(y,\bar{d})\}$$

формульны и разбиваются на конечное число выпуклых множеств. Следовательно, область определения функции $y = f(x, \bar{d})$ состоит из конечного числа выпуклых A-определимых множеств, на каждом из которых существует $\emptyset$-определимая или константная функция, эквивалентная функции $y = f(x, \bar{d})$.

Литература

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ОБ АЛГЕБРАХ УСЛОВНО ТЕРМАЛЬНЫХ ФУНКЦИЙ[1]

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Понятие условного терма сигнатуры σ введено в работе автора [1]. В ряде дальнейших работ [2]-[7] довольно детально исследовались свойства условно термальных функций и были получены результаты, демонстрирующие важность изучения этих функций для исследования строения универсальных классов универсальных алгебр, в том числе, для исследования строения конечных универсальных алгебр. Роль условно термальных функций при изучении универсальных классов (как функций сохраняемых в категории вложимости универсального класса) аналогична роли термальных функций при изучении многообразий (функций сохраняемых в полной категории многообразия). В связи с этим возникает естественный вопрос, в какой мере алгебра условно термальных функций над универсальной алгеброй A играет, в отношении универсального класса порожденного алгеброй роль алгебры термальных над A функций (в частности, алгебры порождающей то же многообразие, что и алгебра A). Этому, а так же ряду близких вопросов относительно алгебр условно термальных функций, посвящена данная работа.

Будут использоваться стандартные обозначения и понятия универсальной алгебры, при этом сигнатуры рассматриваются чисто функциональными и, для простоты, не более чем счетными. Понятия условия, условного терма сигнатуры σ, соответствующей условно термальной функции на алгебре сигнатуры σ, нормальной формы условного терма можно найти в работах [1]-[6]. Пусть $X = \{x_1, \ldots, x_n, \ldots\}$ счетная совокупность переменных. Через $T_n(\sigma)$ ($CT_n(\sigma)$) обозначим совокупность всех термов (условных термов) сигнатуры σ от переменных

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$\{x_1, \ldots, x_n\}$. Пусть $T(\sigma) = \bigcup\limits_{n\in\omega} T_n(\sigma)$ и $CT(\sigma) = \bigcup\limits_{n\in\omega} CT_n(\sigma)$. Имеют место включения $T_n(\sigma) \subseteq CT_n(\sigma)$, $T(\sigma) \subseteq CT(\sigma)$. Каждому условному терму $t(x_{i_1}, \ldots, x_{i_n}) \in CT(\sigma)$ на любой универсальной алгебре $\mathcal{A} =< A; \sigma >$ соответствует условно термальная функция $t_{\mathcal{A}} : A^n \to A$. Если φ — вложение множества $\{x_{i_1}, \ldots, x_{i_n}\}$ в конечное подмножество $S \subseteq X$, то через $t^{\varphi}_{\mathcal{A}}$ обозначим соответствующую функцию из $A^{|S|}$ в A независящую от координат $S \setminus \varphi(\{x_{i_1}, \ldots, x_{i_n}\})$. Пусть Φ — совокупность всех тождественных вложений любых конечных подмножеств множества X в произвольные конечные подмножества множества X и $P_\omega(X)$ — совокупность конечных подмножеств множества X. Для $S \in P_\omega(X)$, через $\mathcal{A}^{CT_S}$ обозначим совокупность всех условно термальных функций от переменных S на алгебре $\mathcal{A}$ рассматриваемую естественным образом, как алгебру сигнатуры σ. Вложение $\varphi \in \Phi$, $\varphi : S_1 \to S_2$ позволяет отождествлять функции $t_{\mathcal{A}}$ и $t^{\varphi}_{\mathcal{A}}$, где t — терм от переменных S_1. Под *алгеброй* $\mathcal{A}^{CT}$ *условно термальных функций алгебры* $\mathcal{A}$ будем понимать универсальную алгебру сигнатуры σ элементами которой являются условно термальные функции от переменных X с учетом отождествления соответствующих функций, связанного с отображениями $\varphi \in \Phi$. При этом алгебры $\mathcal{A}^{CT^{\varphi}_S} =< \{t^{\varphi}_{\mathcal{A}} | t \in \mathcal{A}^{CT_S}\}; \sigma >$, при $S \in P_\omega(X)$, $\varphi \in \Phi$ образуют локальное покрытие алгебры $\mathcal{A}^{CT}$. Пусть $\mathcal{A}^{CT_n} = \mathcal{A}^{CT_{\{x_1, \ldots, x_n\}}}$. Так для любого $\varphi \in \Phi$, $\varphi : \{x_1, \ldots, x_n\} \to X$ алгебры $\mathcal{A}^{CT_n}$ и $\mathcal{A}^{CT^{\varphi}_{\{x_1, \ldots, x_n\}}}$ изоморфны, то, с учетом замеченного выше о локальном покрытии алгебры $\mathcal{A}^{CT}$, для доказательства изоморфизма алгебр $\mathcal{A}^{CT}_1$ и $\mathcal{A}^{CT}_2$, достаточно доказательства изоморфизма алгебр $\mathcal{A}^{CT_n}_1$ и $\mathcal{A}^{CT_n}_2$ для любого $n \in \omega$. Заметим наконец, что селекторные функции $S(x_{i_1}, \ldots, x_{i_m}) = x_i$, где $i \in \{i_1, \ldots, i_m\}$, как функции вида $(x_i)^{\varphi}_{\mathcal{A}}$, где φ — вложение $\{x_i\}$ в $\{x_{i_1}, \ldots, x_{i_m}\}$ такое, что $\varphi(x_i) = x_i$, входят в алгебру $\mathcal{A}^{CT}$.

Пусть $\mathcal{T}_n$ совокупность всех полных систем условий сигнатуры σ от переменных $\bar{x} = \{x_1, \ldots, x_n\}$. На $\mathcal{T}_n$ определим порядок $\leq$ следующим образом:

если $T_1(\bar{x}) = \{\varphi^1_1(\bar{x}), \ldots, \varphi^k_1(\bar{x})\}$, $T_2(\bar{x}) = \{\varphi^2_1(\bar{x}), \ldots, \varphi^l_2(\bar{x})\}$ элементы $\mathcal{T}_n$, то положим $T_1(\bar{x}) \leq T_2(\bar{x})$ тогда и только тогда, когда $\forall i \leq l \exists j \leq k$ такое, что $\vdash \forall \bar{x}(\varphi^i_2(\bar{x}) \to \varphi^j_1(\bar{x}))$. Очевидно, что $< \mathcal{T}_n; \leq >$ направленное

множество. Для любой алгебры $\mathcal{A}$, любого $T(\bar{x}) \in \mathcal{T}_n$ через $CT_{T(\bar{x})}(\mathcal{A})$ обозначим совокупность условно термальных функций из $\mathcal{A}^{CT_n}$ таких, что полная система условий для нормальных форм (см. [1]) этих функций равна $T(\bar{x})$. Для $T_1(\bar{x}) \leq T_2(\bar{x})$ существует естественное вложение алгебры $CT_{T_1(\bar{x})}(\mathcal{A})$ в алгебру $CT_{T_2(\bar{x})}(\mathcal{A})$ (любая функция из $CT_{T_1(\bar{x})}(\mathcal{A})$ естественным образом рассматривается как функция из $CT_{T_2(\bar{x})}(\mathcal{A})$). Таким образом, алгебра $\mathcal{A}^{CT_n}$ представима как прямой предел алгебр $CT_{T(\bar{x})}(\mathcal{A})$ относительно направленного множества $< \mathcal{T}_n; \leq >$:

$$\mathcal{A}^{CT_n} = \lim_{T(x) \in < \mathcal{T}_n; \leq >} CT_{T(\bar{x})}(\mathcal{A}) \tag{1}$$

Через $\mathcal{A}^{T_n}$, $\mathcal{A}^{T}$ и т.д. обозначим *алгебры термальных функций на алгебре* $\mathcal{A}$ (с заменой в соответствующих определениях условно термальных функций на термальные). Таким образом $\mathcal{A}^{T_n} \cong \mathcal{F}_{\mathcal{M}(\mathcal{A})^{(n)}}$, $\mathcal{A}^{T} \cong \mathcal{F}_{\mathcal{M}(\mathcal{A})^{(\omega)}}$, где $\mathcal{M}(\mathcal{A})$ — многообразие порожденное алгеброй $\mathcal{A}$, а $\mathcal{F}_{\mathcal{M}(\mathcal{A})^{(n)}}$, $\mathcal{F}_{\mathcal{M}(\mathcal{A})^{(\omega)}}$ — свободные n- и счетно порожденные алгебры многообразия $\mathcal{M}(\mathcal{A})$.

Пусть $\varphi(\bar{x}) = \varphi(x_1, \ldots, x_n)$ некоторое условие сигнатуры σ. Для прямой степени $\mathcal{A}^{(A^n)}$ через π_φ обозначим проектирование алгебры $\mathcal{A}^{(A^n)}$ на множеству $\{\bar{a} \in A^n | \mathcal{A} \models \varphi(\bar{a})\}$. Для полной системы условий $T(\bar{x}) = \{\varphi_1(\bar{x}), \ldots, \varphi_m(\bar{x})\} \in \mathcal{T}_n$ очевидно равенство

$$CT_{T(\bar{(x)})}(\mathcal{A}) = \pi_{\varphi_1}(\mathcal{A}^{T_n}) \times \cdots \times \pi_{\varphi_m}(\mathcal{A}^{T_n}). \tag{2}$$

Для алгебр $\mathcal{A}_1$ и $\mathcal{A}_2$ через $\mathcal{A}_1 \equiv_\forall \mathcal{A}_2$ обозначим совпадение универсальных теорий (т.е. $\forall$-эквивалентность) алгебр $\mathcal{A}_1$ и $\mathcal{A}_2$. Имеет место:

ТЕОРЕМА 1. Для любых $\forall$-эквивалентных алгебр $\mathcal{A}_1$ и $\mathcal{A}_2$ алгебры $\mathcal{A}^{CT}_1$ и $\mathcal{A}^{CT}_2$ изоморфны.

ДОКАЗАТЕЛЬСТВО. Действительно, пусть $\mathcal{A} \equiv_\forall \mathcal{A}_2$. Тогда для любого $n \in \omega$, любой полной системы условий $T(\bar{x}) = \{\varphi_1(\bar{x}), \ldots, \varphi_m(\bar{x})\} \in \mathcal{T}_n$ и любых термов $t_1(\bar{x})$, $t_2(\bar{x})$ сигнатуры σ имеет место

$$\forall i \leq m(\mathcal{A}_1 \models \forall \bar{x}(\varphi_i(\bar{x}) \to t_1(\bar{x}) = t_2(\bar{x})) \Leftrightarrow \mathcal{A}_2 \models \forall \bar{x}(\varphi_i(\bar{x}) \to t_1(\bar{x}) = t_2(\bar{x}))).$$

Таким образом, для любого $i \leq n$

$$\pi_{\varphi_i}(\mathcal{A}^{T_n}_1) \cong \pi_{\varphi_i}(\mathcal{A}^{T_n}_2).$$

В силу формулы (2), $CT_{T(\overline{x})}(\mathcal{A}_1) = CT_{T(\overline{x})}(\mathcal{A}_2)$.
А в силу формулы (1),

$$\mathcal{A}_1^{CT_n} \cong \mathcal{A}_2^{CT_n}.$$

Как замечено выше, этого достаточно для доказательства изоморфизма

$$\mathcal{A}_1^{CT} \cong \mathcal{A}_2^{CT}.$$

Из указанной выше аналогии: универсальные классы алгебр — многообразия, алгебра условно термальных функций — алгебра термальных функций (т.е. свободная алгебра многообразия), можно было бы ожидать справедливости обратного к теореме 1. утверждения. Однако следующий простой пример опровергает это.

ПРИМЕР 1. Пусть сигнатура σ состоит из одной одноместной функции f. Алгебра $\mathcal{A}_1$ является дизъюнктным объединением циклов (относительно f) длины 3 и 2, а алгебра $\mathcal{A}_2$ — цикл длины 6. Таким образом, $\mathcal{A}_1 \not\equiv_\forall \mathcal{A}_2$. С другой стороны, т.к. $\mathcal{A}_i^{CT_n} \subseteq \mathcal{A}_i^{(\mathcal{A}_i^n)}$ и $\mathcal{A}_i \models \forall x(f^6(x) = x)$, то алгебры $\mathcal{A}_i^{CT_n}$ (при $n \in \omega$) представляют собой дизъюнктные объединения f-циклов длины 6, 3, 2. Очевидно, что f-циклов длины 3, 2 в этом представлении быть не может, т.к. таких циклов нет в алгебре $\mathcal{A}_2$, а для любой условно термальной функции $g(x_1, \ldots, x_n)$ над алгеброй $\mathcal{A}_1$, если $a_1, \ldots, a_n$ входят в цикл длины 2 (3), то и $g(a_1, \ldots, a_n)$ входит в этот же цикл и, значит, сама функция $g(x_1, \ldots, x_n) \in \mathcal{A}_1^{CT_n}$ входит в цикл длины 6. Таким образом, счетные алгебры $\mathcal{A}_i^{CT}$ допускают локальное покрытие дизъюнктными объединениями f-циклов длины 6 и, значит, $\mathcal{A}_1^{CT} \cong \mathcal{A}_2^{CT}$.

С другой стороны этот пример показывает, что в отличии от алгебр термальных функций, где изоморфизм $\mathcal{A}_1^T \cong \mathcal{A}_2^T$ ($\mathcal{F}_{\mathcal{M}(\mathcal{A}_1)}(\omega) \cong \mathcal{F}_{\mathcal{M}(\mathcal{A}_2)}(\omega)$) влечет существование изоморфизмов $\mathcal{A}_1^{T_n} \cong \mathcal{A}_2^{T_n}$ ($\mathcal{F}_{\mathcal{M}(\mathcal{A}_1)}(n) \cong \mathcal{F}_{\mathcal{M}(\mathcal{A}_2)}(n)$) для алгебр условно термальных функций это не так. В рассмотренном примере $\mathcal{A}_1^{CT} \cong \mathcal{A}_2^{CT}$. В то же время непосредственные вычисления показывают, что $\mathcal{A}_1^{CT_1} \cong \mathcal{A}_2^{CT_1} \cong \mathcal{A}_2$, но $|\mathcal{A}_1^{CT_2}| = 540$, $|\mathcal{A}_2^{CT_2}| = 6^6$, т.е. $\mathcal{A}_1^{CT_2} \not\cong \mathcal{A}_2^{CT_2}$.

Из этого же примера в частности следует, что, вообще говоря, $\mathcal{A} \not\equiv_\forall \mathcal{A}^{CT}$ (в отличии от равенства $\mathcal{M}(\mathcal{A}) = \mathcal{M}(\mathcal{A}^T)$).

Правда в ряде случаев можно утверждать, что любая $\forall$-формула истинная на алгебре $\mathcal{A}^{CT}$ будет истинна на алгебре $\mathcal{A}$.

Алгебру $\mathcal{A}$ назовем *устойчивой*, если либо она идемпотентна, либо существует терм $t(x)$ такой, что $\mathcal{A} \models \forall x, y(t(x) = t(y))$ и $\{t(a)\}$ является подалгеброй алгебры $\mathcal{A}$ для любого $a \in \mathcal{A}$.

Устойчивыми, в частности являются группы, кольца,решетки.

ТЕОРЕМА 2. Для любой устойчивой алгебры $\mathcal{A}$, любой $\exists$-формулы φ, если $\mathcal{A} \models \varphi$, то $\mathcal{A}^{CT} \models \varphi$.

Доказательство. Пусть $\varphi = \exists \overline{x}\, \psi(\overline{x}) = \exists x_1, \ldots, x_n\, \psi(x_1, \ldots, x_n)$, где $\psi(\overline{x})$ бесквантерная формула. Можно считать, что $\psi(\overline{x})$ — условие. Через $s^i(\overline{x})$ обозначим селекторы для $i \leq n$ ($s^i(x_1, \ldots, x_n) = x_i$). Тогда для любого терма $q(\overline{x})$ сигнатуры σ, для любых $\overline{a} \in \mathcal{A}$ имеет место равенство

$$q(\overline{a}) = q(s^1, \ldots, s^n)(\overline{a}).$$

Рассмотрим условные термы $s^i_\psi(\overline{x})$ такие, что

$$s^i_\psi(\overline{x}) = \begin{cases} \psi(\overline{x}) & \to \; s^i(\overline{x}) \\ \neg\psi(\overline{x}) & \to \; t(s^1(\overline{x})), \end{cases}$$

где $t(\overline{x})$ — терм из определения устойчивости алгебры $\mathcal{A}$ в случае её неидемпотентности или

$$s^i_\psi(\overline{x}) = \begin{cases} \psi(\overline{x}) & \to \; s^i(\overline{x}) \\ \neg\psi(\overline{x}) & \to \; s^1(\overline{x}), \end{cases}$$

если алгебра $\mathcal{A}$ идемпотентна.

Непосредственно проверяется, что $\mathcal{A} \models \exists\overline{x}\psi(\overline{x}) \Rightarrow \mathcal{A}^{CT} \models \psi(s^1_\psi, \ldots, s^n_\psi)$, т.е. $\mathcal{A}^{CT} \models \exists\overline{x}\psi(\overline{x})$.

В частности, если φ некоторая $\forall$-формула и $\mathcal{A}$ устойчивая алгебра, то

$$\mathcal{A}^{CT} \models \varphi \Rightarrow \mathcal{A} \models \varphi.$$

Пример 1 демонстрирует так же то, что для произвольных алгебр утверждение теоремы 2 не верно:

$$\mathcal{A}_1 \models \exists x(f^3(x) = x),\ \mathcal{A}_1^{CT} \not\models \exists x(f^3(x) = x).$$

Заметим, что, т.к. $\mathcal{A}^{CT_n}$ является подалгеброй алгебры $\mathcal{A}^{(\mathcal{A}^n)}$ и алгебры изоморфные алгебрам $\mathcal{A}^{CT_n}$ образуют локальное покрытие алгебры $\mathcal{A}^{CT}$, то для любого квазитождества φ

$$\mathcal{A} \models \varphi \Rightarrow \mathcal{A}^{CT} \models \varphi$$

Через $Q(\mathcal{A})$ обозначим квазимногообразие порожденное алгеброй $\mathcal{A}$. В силу теоремы 2 и замеченного вслед за её доказательством, имеет место

СЛЕДСТВИЕ 1. Для любой устойчивой алгебры $\mathcal{A}$ справедливо равенство $Q(\mathcal{A}) = Q(\mathcal{A}^{CT})$.

Следствие 1 и теорема 1 влекут естественный вопрос: нельзя ли усилить утверждение теоремы 1 следующим образом

$$Q(\mathcal{A}_1) = Q(\mathcal{A}_2) \Rightarrow \mathcal{A}_1^{CT} \cong \mathcal{A}_2^{CT}?$$

Отрицательный ответ на этот вопрос вытекает из следующего примера.

ПРИМЕР 2. Пусть $\mathcal{A}_1$ двухэлементная, а $\mathcal{A}_2$ — трехэлементная решетки. Таким образом, действительно имеет место равенство $Q(\mathcal{A}_1) = Q(\mathcal{A}_2)$. Покажем, что $\mathcal{A}_1^{CT} \not\cong \mathcal{A}_2^{CT}$. Через $\mathcal{A}^d$ обозначим обогащение произвольной алгебры $\mathcal{A}$ введением в сигнатуру нового трехместного функционального символа $d(x,y,z)$ интерпретируемого как дискриминатор на $\mathcal{A}$. В работе [1] отмечено, что условно термальные функции алгебры $\mathcal{A}$ суть термальные функции алгебры $\mathcal{A}^d$. Таким образом $\mathcal{A}^{CT} \cong ((\mathcal{A}^d)^T)' \cong (\mathcal{F}_{\mathcal{M}(\mathcal{A}^d)}(\omega))'$, где $'$ означает обеднение указанной алгебры до исходной сигнатуры (удалением из сигнатуры функции d). Из известных описаний свободных алгебр конечно порожденных дискриминаторных многообразий (см., к примеру [8]), $\mathcal{F}_{\mathcal{M}(\mathcal{A}_1^d)}(\omega)$ и $\mathcal{F}_{\mathcal{M}(\mathcal{A}_2^d)}(\omega)$ представимы булевыми произведениями простых алгебр многообразий $\mathcal{M}(\mathcal{A}_1)$ и $\mathcal{M}(\mathcal{A}_2)$, т.е. алгебр $\mathcal{A}_1^d$ и одноэлементной алгебры в первом случае и алгебр $\mathcal{A}_1^d$, $\mathcal{A}_2^d$, одноэлементной — во втором. Наличие сомножителя вида $\mathcal{A}_2'$ в представлении алгебры $(\mathcal{F}_{\mathcal{M}(\mathcal{A}_2^d)}(\omega))'$ означает существование конгруэнции Θ на этой алгебре фактор по которой трехэлементен. Но тогда изоморфизм $\mathcal{A}_1^{CT} \cong \mathcal{A}_2^{CT}$, т.е. $(\mathcal{F}_{\mathcal{M}(\mathcal{A}_1^d)}(\omega))' \cong (\mathcal{F}_{\mathcal{M}(\mathcal{A}_2^d)}(\omega))'$, повлечет существование на алгебре $(\mathcal{F}_{\mathcal{M}(\mathcal{A}_1^d)}(\omega))'$, булевом произведении двухэлементных решеток, такой конгруэнции, фактор по которой трехэлементен. В виду конгруэнц-дистрибутивности решеток, а значит, сводимости конгруэнций на булевом произведении $(\mathcal{F}_{\mathcal{M}(\mathcal{A}_1^d)}(\omega))'$ к ядрам проэктирований (см., к примеру, [8]), это невозможно. А, значит, невозможен и изморфизм $\mathcal{A}_1^{CT} \cong \mathcal{A}_2^{CT}$.

Отметим так же следующий достаточно очевидный факт. Определение фрагмента L_{HYP} логики $L_{\omega_1,\omega}$ см., к примеру, [9].

ТЕОРЕМА 3. Для любой формулы φ узкого исчисления предикатов существует формула $\varphi' \in L_{HYP}$ такая, что для любой алгебры $\mathcal{A}$ имеет место эквивалентность

$$\mathcal{A}^{CT} \models \varphi \Leftrightarrow \mathcal{A} \models \varphi'.$$

ДОКАЗАТЕЛЬСТВО. Ограничимся, для примера, построением формулы φ для формулы $\varphi = \exists x_1, \ldots, x_k(\&_{i=1}^{l} t_i^1(\overline{x}) = t_i^2(\overline{x}) \& \&_{j=1}^{m} q_j^1(\overline{x}) \neq q_j^2(\overline{x}))$, где t_i^s, q_j^s — термы рассматриваемой сигнатуры.

В данном случае полагаем

$$\varphi' = \bigvee_{n\in\omega} \bigvee_{\langle\psi_1(\overline{y}),\ldots,\psi_p(\overline{y})\rangle\in T_n} \bigvee_{<r_1^1(\overline{y}),\ldots,r_p^1(\overline{y})>\in(T_n)^p} \cdots \bigvee_{<r_1^k(\overline{y}),\text{\textbackslash ldots},r_p^k(\overline{y})>\in(T_n)^p}$$

$$[\forall\overline{y}(\&_{s=1}^{p}(\psi_s(\overline{y}) \to \&_{i=1}^{l} t_i^1(r_s^1(\overline{y}),\ldots,r_s^k(\overline{y})) = t_i^2(r_s^1(\overline{y}),\ldots,r_s^k(\overline{y}))))\&$$

$$\&_{j=1}^{m}(\vee_{s=1}^{p}\exists\overline{y}_s(\psi_s(\overline{y}_s)\&q_j^1(r_s^1(\overline{y}_s),\ldots,r_s^k(\overline{y}_s)) \neq q_j^2(r_s^1(\overline{y}_s),\ldots,r_s^k(\overline{y}_s)))))].$$

Утверждение $\mathcal{A}^{CT} \models \varphi \Leftrightarrow \mathcal{A} \models \varphi'$ очевидно.

Заметим, что на самом деле L_{HYP}-формула ψ' является замыканием совокупности $\forall$- и $\exists$-формул относительно бесконечных (HYP-) конъюнкций и дизъюнкций.

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ВНУТРЕННИЕ ИЗОМОРФИЗМЫ И УСЛОВНО РАЦИОНАЛЬНАЯ ЭКВИВАЛЕНТНОСТЬ УНАРАМ И ПОЛЯМ[1]

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В работах [1], [2] автора были введены понятия условного терма и условно рациональной эквивалентности универсальных алгебр, позволившие получить некоторую новую классификацию универсальных алгебр по их вычислительным возможностям. Инвариантами этой классификации выступают инверсные полугруппы внутренних изоморфизмов универсальных алгебр. Напомним, что внутренним изоморфизмом алгебры $\mathcal{A}$ называется изоморфизм между подалгебрами алгебры $\mathcal{A}$. Совокупность всех внутренних изоморфизмов алгебры $\mathcal{A}$ (включая и пустое отображение) образует, относительно стандартным образом определенной композиции, инверсную полугруппу обозначенную далее как $\mathrm{Iso}\,\mathcal{A}$. Идемпотенты этой полугруппы естественным образом ассоциируются с подалгебрами алгебры $\mathcal{A}$. В работе [2], в частности, были доказаны следующие утверждения.

ТЕОРЕМА A. Две конечные универсальные алгебры $\mathcal{A} =< A; \sigma_1 >$ и $\mathcal{L} =< B, \sigma_2 >$ условно рационально эквивалентны (обладают равными вычислительными возможностями) тогда и только тогда, когда существует взаимно однозначное отображение π множества A на множество B такое, что

$$\pi^{-1}\mathrm{Iso}\mathcal{L}\pi = \mathrm{Iso}\mathcal{A}.$$

ТЕОРЕМА B. Для любых конечных универсальных алгебр $\mathcal{A}$ и $\mathcal{L}$ инверсные полугруппы $< \mathrm{Iso}\mathcal{A}, U_{\mathcal{A}} >$ и $< \mathrm{Iso}\mathcal{L}, U_{\mathcal{L}} >$ (обогащенные предикатами U, выделяющими идемпотенты соответствующие одноэлементным

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подалгебрам) изоморфны тогда и только тогда, когда существует натуральное n, обратимый идемпотентный терм $\eta(x)$ сигнатуры алгебры $\mathcal{A}^{[n]}$ такие что алгебры $\mathcal{A}^{[n]}(\eta)$ и $\mathcal{L}$ условно рационально эквивалентны.

Здесь $\mathcal{A}^{[n]}$ — матричная степень алгебры $\mathcal{A}$ и $\mathcal{A}^{[n]}(\eta)$ — η-редукт алгебры $\mathcal{A}^{[n]}$. Соответствующие определения можно найти, к примеру, в [3]. Алгебры $\mathcal{A}$ и $\mathcal{L}$ связанные между собой так как это описано в теореме B называются схожими. Таким образом, полугруппы $\operatorname{Iso}\mathcal{A}$ (их конкретное представление) выступают как инварианты отношения условно рациональной эквивалентности на конечных алгебрах, а типы изоморфизма пар $< \operatorname{Iso}\mathcal{A}, U_{\mathcal{A}} >$ выступают инвариантами отношения схожести между конечными алгебрами.

В работах [4], [5] автора на основе этих инвариантов описаны условия для того, что бы конечная алгебра $\mathcal{A}$ была условно рационально эквивалентна некоторой полурешетке, решетке, булевой алгебре. В настоящей работе подобные условия найдены для того что бы алгебра $\mathcal{A}$ была условно рациональна эквивалентна (схожа с) некоторому унару, моноунару, конечному полю. Как всегда под унарами (моноунарами) понимается универсальная алгебра сигнатура которой состоит лишь из (одной) одноместных функций.

В работе [2] описаны условия для представления инверсной полугруппы (для инверсной полугруппы с выделенным множеством идемпотентов) частичных биекций на множестве A в виде $\operatorname{Iso}\mathcal{A}$ для какой либо алгебры $\mathcal{A} =< A, \sigma >$ (в виде пары изоморфной паре $< \operatorname{Iso}\mathcal{A}, U_{\mathcal{A}} >$ для некоторой алгебры $\mathcal{A}$). А именно доказаны два следующих утверждения.

ТЕОРЕМА C. Для пары $< H, S >$, где H — некоторая совокупность подмножеств множества A и S — совокупность биекций между подмножествами из H, существует универсальная алгебра $\mathcal{A} =< A, \sigma >$ с условиями $H = Sub\,\mathcal{A}$, $S = \operatorname{Iso}\mathcal{A}$ тогда и только тогда, когда H — алгебраическая решетка подмножеств включающая A и $\emptyset$ и пара $< H, S >$ удовлетворяет принципам обратимости,композиции, неподвижных точек, ограничения, согласованности, глобализации и одноэлементных подалгебр.

Здесь под решеткой подмножеств множества A понимается решеточно упорядоченная совокупность подмножеств множества A для которой роль операции $\wedge$ играет теоретико-множественное пересечение, а выше перечисленные принципы таковы:

а) принцип обратимости: для $g \in S$ отображение g^{-1} так же входит в S;

б) принцип композиции: для $g, h \in S$ если $r_h = d_g$, то $g \circ h \in S$ (d_h — область определения h, r_h — область значения h);

в) принцип неподвижных точек: для $g \in S$ множество $\operatorname{Fix} g = \{a \in A \mid g(a) = a\}$ входит в H;

г) принцип ограничения: для $g \in S$ и $C \in H$ если $C \subset d_g$, то $g|_C \in S$ ($g|_C$ — ограничение g до C);

д) принцип согласованности: для $C \in H$ $id_C \in S$ (id_C — тождественное отображение на C), для $g \in S$ $d_g, r_g \in H$;

е) принцип глобализации: для любого $C \in H$ и любого отображения $F : P_\omega(C) \to S$ такого, что

1) для $D \in P_\omega(C)$ $d_{F(D)} = H(D)$ и

2) для любых $D_1, D_2 \in P_\omega(C)$

$$F(D_1)|_{H(D_1)\cap H(D_2)} = F(D_2)|_{H(D_1)\cap H(D_2)}$$

существует $h \in S$ такое, что $d_h = C$ и для любого $D \in P_\omega(C)$ $h|_{H(D)} = F(D)$. Здесь $P_\omega(C)$ — совокупность всех конечных подмножеств множества C, а $H(D)$ наименьшее подмножество из H содержащее D;

ж) принцип одноэлементных подалгебр: для любых $a, b \in A$ если $\{a\}, \{b\} \in H$, то существует $h \in S$ такое, что $d_h = \{a\}$, $r_h = \{b\}$.

ТЕОРЕМА D. Пусть $\mathcal{L}$инверсная полугруппа и U некоторое подмножество её идемпотентов. Тогда существование универсальной алгебры $\mathcal{A}$ такой, что $< \mathcal{L}, U > \cong < \operatorname{Iso}\mathcal{A}, U_{\mathcal{A}} >$ эквивалентно следующим условиям: совокупность идемпотентов полугруппы $\mathcal{L}$ образует алгебраическую решетку с нулем 0 и единицей 1, U — замкнутое подмножество строго минимальных идемпотентов полугруппы $\mathcal{L}$ и выполнены условия а')-в').

Здесь ненулевой идемпотент e полугруппы $\mathcal{L}$ называется строго минимальным, если e минимален относительно стандартного порядка на ненулевых идемпотентах и для любого $g \in \mathcal{L}$ из равенства $gg^{-1} = g^{-1}g = e$

вытекает равенство $g = e$. Подмножество U идемпотентов называется замкнутым, если для любых $e \in U$, $g \in \mathcal{L}$ либо $g^{-1}eg = 0$, либо $g^{-1}eg \in U$.

Условие a′): для любого $g \in U \setminus 0$ существует идемпотент e такой, что равенство $bg^{-1}g = bg^{-1}$ равносильно равенству $bg^{-1} = be$.

Условие б′): для любого идемпотента e и любого отображения F : $C(e) = \{e' | e'$ — идемпотент и $e' \leq e\} \to \mathcal{L}$ такого, что

1) для $h \in C(e)$ $h = F(h)^{-1}F(h)$;

2) для $h_1, h_2 \in C(e)$ $F(h_1)h_1h_2 = F(h_2)h_1h_2$

найдется $f \in \mathcal{L}$ такой, что $f^{-1}f = e$ и $F(h)h = fh$ для любого $h \in C(e)$.

Условие в′): для $e_1, e_2 \in U$ существует $g \in \mathcal{L}$ такой, что $g^{-1}g = e_1$ и $gg^{-1} = e_2$.

В дальнейшем пары $< H, S >$ удовлетворяющие условиям теоремы C и пары $< \mathcal{L}, U >$ удовлетворяющие условиям теоремы D назовем соответственно конкретно алгебраически допустимыми и абстрактно алгебраически допустимыми.

Перейдем теперь к описанию инвариантов отношений условно рациональной эквивалентности и схожести для унаров и моноунаров.

ТЕОРЕМА 1. Для конкретно алгебраически допустимой пары $< H, S >$ на множестве A найдется унар $\mathcal{A} =< A; \sigma >$ такой, что $H = Sub\,\mathcal{A}$, $S = \mathrm{Iso}\,\mathcal{A}$ в том и только том случае когда решетка H дистрибутивна, для любого $a \in A$ элемент $H(\{a\})$ $\vee$-неразложим в решетке H и S обладает свойством амальгамирования: для $g_1, g_2 \in S$, если $g_1|_{d_{g_1} \cap d_{g_2}} = g_2|_{d_{g_1} \cap d_{g_2}}$, то существует $h \in S$ такое, что $h|_{d_{g_1}} = g_1$ и $h|_{d_{g_2}} = g_2$. Верно и обратное: для любого унара $\mathcal{A}$ пара $< H(\mathcal{A}), S(\mathcal{A}) >$ обладает всеми указанными здесь свойствами пары $< H, S >$.

Доказательство. Доказательство основано на конструкции восходящей к Йонссону и использованной в доказательствах теорем Стоуна [6], Бредихина [7] и теоремы C.

Для любого $a \in A$ и любого $a' \in H(\{a\})$ введем в рассмотрение одноместную функцию $f_{a,a'}(x)$ на A следующим образом:

$$f_{a,a'}(b) = \begin{cases} g(a'), & \text{если существует } g \in S \\ & \text{такое, что } g(a) = b; \\ b & \text{в противном случае.} \end{cases}$$

Прежде всего отметим корректность определения функции $f_{a,a'}$ на множестве A. Пусть $g_1, g_2 \in S$ таковы, что $b = g_1(a) = g_2(a)$. Покажем, что $g_1(a') = g_2(a')$. По принципу ограничения найдутся $h_1, h_2 \in S$ такие, что $h_i = g_i|_{H(\{a\})}$. При этом $h_1(H(\{a\})) = h_2(H(\{a\})) \supseteq \{b\}$ и, следовательно, $h_1(H(\{a\})) \cap h_2(H(\{a\})) \supseteq H(\{b\})$. По принципам ограничения и обратимости найдется $f \in S$ такое, что $f = h_1^{-1}|_{H(\{b\})}$. По принципу композиции $r_{fh_2} = H(\{a\})$ и $fh_2(a) = a$. По принципу неподвижных точек $\mathrm{Fix}\, fh_2 = H(\{a\})$ и, значит, $fh_2(a') = a'$, т.е. $f^{-1}(a') = h_2(a')$. Но последнее означает, что $h_1(a') = h_2(a')$ и, тем самым, $g_1(a') = g_2(a')$. Таким образом, функции вида $f_{a,a'}$ определены на A корректно.

Пусть $\mathcal{A} =< A; f_{a,a'}|$ где $a \in A$ и $a' \in H(\{a\}) >$. Очевидно, что совокупность $H_1 = \{H(\{a\}) | a \in A\}$ совпадает с совокупностью $Sub_1\,\mathcal{A}$ — однопорожденных подалгебр алгебры $\mathcal{A}$. Покажем, что $H = Sub\,\mathcal{A}$. Достаточно показать совпадение $H_\omega = \{H(\{a_1, \ldots, a_n\}) | n \in \omega$ и $a_1, \ldots, a_n \in A\}$ и совокупности $Sub_\omega\mathcal{A}$ всех конечно порожденных подалгебр алгебры $\mathcal{A}$. Для этого, в свою очередь, покажем, что для любых $a_1, \ldots, a_n \in A$ подалгебра $< a_1, \ldots, a_n >$ алгебры $\mathcal{A}$ порожденная множеством $\{a_1, \ldots, a_n\}$ совпадает с $H(\{a_1, \ldots, a_n\})$. Включение $< a_1, \ldots, a_n >= \bigcup\limits_{i=1}^{n} < a_i >= \bigcup\limits_{i=1}^{n} H(\{a_i\}) \subseteq H(\{a_1, \ldots, a_n\})$ очевидно. Покажем обратное включение. Пусть $c \in H(\{a_1, \ldots, a_n\})$. Тогда $H(\{c\}) \subseteq H(\{a_1, \ldots, a_n\})$. Но $H(\{a_1, \ldots, a_n\}) = \bigvee\limits_{i=1}^{n} H(\{a_i\})$ (где $\vee$ — решеточный, в решетке H, супремум). Так как H дистрибутивная решетка, $H(\{c\}) = H(\{c\}) \wedge \left(\bigvee\limits_{i=1}^{n} H(\{a_i\})\right) = \bigvee\limits_{i=1}^{n}\left(H(\{c\}) \cap H(\{a_i\})\right)$ и, значит, так как $H(\{c\})$ $\vee$-неразложим, найдется $j \leq n$ такое, что $H(\{c\}) = H(\{c\}) \cap H(\{a_j\})$, т.е. $c \in H(\{a_j\}) =< a_j > \subseteq < a_1, \ldots, a_n >$. Итак, равенство $H = Sub\,\mathcal{A}$ доказано.

Покажем теперь, что совокупность изоморфизмов между подалгебрами алгебры $\mathcal{A}$ совпадает с S. Пусть $B, C \in H$ и h — изоморфизм между подалгебрами $\mathcal{L}$ и $\mathcal{V}$ алгебры $\mathcal{A}$ с основными множествами B и C соответственно. Отображение $F : P_\omega(B) \to S$ определим следующим образом: для $D \in P_\omega(B)$ пусть $F(D) = h|_{H(D)}$. Для доказательства включения $h \in S$ достаточно в силу принципа глобализации показать, что для любого $D \in P_\omega(B)$ отображение $h_D = h|_{H(D)}$ принадлежит S. Пусть $D = \{a_1, \ldots, a_n\}$. В силу доказанного выше $H(D) = \bigcup_{i=1}^{n} H(\{a_i\})$ и, тем самым, по свойству амальгамируемости для S, достаточно заметить, что $h_i = h|_{H(\{a_i\})} \in S$ для любого $i \leq n$. Пусть $i = 1$. Если $|H(\{a_1\})| \neq 1$, то пусть $a' \neq a$ и $a' \in H(\{a_1\})$. Тогда $f_{a_1,a'}(a_1) = a'$ и, значит, $f_{a_1,a'}(h_1(a_1)) = h_1(a')$ и $h_1(a_1) \neq h_1(a')$. Тем самым, по построению функции $f_{a_1,a'}$, найдется $g \in S$ такое, что $g(a_1) = h_1(a_1)$ и $g(a') = h_1(a')$. По принципу ограничения, $g|_{H(\{a_1\})} = h_1$, т.е. $h_1 \in S$. Если же $|H(\{a_1\})| = 1$, то $h_1 \in S$ в силу принципа одноэлементных подалгебр. Таким образом, любой изоморфизм между подалгебрами алгебры $\mathcal{A}$ принадлежит S. Обратное, т.е. то что отображения из S являются изоморфизмами между подалгебрами алгебры $\mathcal{A}$ очевидно в силу определения функции $f_{a,a'}$. Так же очевидно, что для любого унара $\mathcal{A}$ пара $< H(\mathcal{A}), S(\mathcal{A}) >$ обладает всеми свойствами указанными в формулировке теоремы для пары $< H, S >$. Теорема доказана.

СЛЕДСТВИЕ 1. Конечная алгебра $\mathcal{A}$ условно рационально эквивалентна некоторому унару тогда и только тогда, когда решетка $Sub\,\mathcal{A}$ дистрибутивна, однопорожденные подалгебры алгебры $\mathcal{A}$ $\vee$-неразложимые в решетке $Sub\,\mathcal{A}$ и полугруппа $\mathrm{Iso}\,\mathcal{A}$ обладает свойством амальгамируемости.

Рассмотрим теперь абстрактную характеризацию пар $< \mathcal{L}, U >$ изоморфных парам $< \mathrm{Iso}\,\mathcal{A}, U_\mathcal{A} >$ для унаров $\mathcal{A}$. Наряду с приведенными вслед за формулировкой теоремы D условий а′),б′),в′) сформулируем и следующие условия на пару $< \mathcal{L}, U >$ где U некоторое замкнутое подмножество строго минимальных идемпотентов инверсной полугруппы $\mathcal{L}$, совокупность идемпотентов которой образует решетку:

г′): Для любых $g_1, g_2 \in \mathcal{L}$ таких, что для произвольных $h_1, h_2 \in \mathcal{L}$ из равенства $h_1 g_1 g_1^{-1} = h_2 g_2 g_2^{-1}$ вытекает равенство $h_1 g_1 = h_2 g_2$, найдется h такой, что для любого $h_1 \in \mathcal{L}$ имеют место равенства $h_1 g_1 g_1^{-1} h = h_1 g_1$ и $h_1 g_2 g_2^{-1} h = h_1 g_2$.

д′): Для любого $g \in \mathcal{L}$ удовлетворяющего условию:

(*) существует $h \in \mathcal{L}$ такой, что для любых $c, b, t \in \mathcal{L}$ таких, что

$hgg^{-1} = bcc^{-1}$ найдется $r \in \mathcal{L}$ удовлетворяющий равенству $tgg^{-1} = rcc^{-1}$, элемент gg^{-1} является $\vee$-неразложимым в решетке идемпотентов полугруппы $\mathcal{L}$.

ТЕОРЕМА 2. Пусть $\mathcal{L}$ инверсная полугруппа, U некоторая замкнутая совокупность её строго минимальных идемпотентов. Тогда следующие условия эквивалентны:

1) для некоторого унара $\mathcal{A}$ пары $< \mathcal{L}, U >$ и $< \mathrm{Iso}\,\mathcal{A}, U_\mathcal{A} >$ изоморфны,

2) пара $< \mathcal{L}, U >$ удовлетворяет условиям а′)-д′) и алгебраическая решетка идемпотентов полугруппы $\mathcal{L}$ дистрибутивна, обладает нулем и единицей.

Доказательство. Импликация 1)→2) очевидна. Покажем, что из 2) следует 1). В работе [2] в доказательство теоремы Вагнера-Престона о том, что любая инверсная полугруппа представима полугруппой биекций между подмножествами некоторого множества A, внесены корректировки, позволяющие считать, что в этом представлении идемпотентам входящим в совокупность U и только им соответствуют одноэлементные области определения рассматриваемых биекций. Пусть S — совокупность биекций соответствующих элементам полугруппы $\mathcal{L}$ и H — множество областей определения этих биекций. Условия а′)-в′) для пары $< \mathcal{L}, U >$ и алгебраичность решетки идемпотентов полугруппы $\mathcal{L}$ обладающей нулем и единицей (как замечено в [2]) влекут выполнимость для пары $< H, S >$ условий теоремы A. Из доказательства теоремы Вагнера-Престона вытекает, что в качестве множества A (биекциями между подмножествами которого интерпретируются элементы полугруппы $\mathcal{L}$) выступает множество $\mathcal{L}$ и элемент $g \in \mathcal{L}$ интерпретируется при этом отображением φ_g с областью определения $\mathcal{L}gg^{-1}$, действующим на $\mathcal{L}gg^{-1}$ следующим образом: для $a \in \mathcal{L}gg^{-1}$ $\varphi_g(a) = ag$. Очевидно, что тогда условие д′) соответствует требованию $\vee$-неразложимости в решетке H элементов вида $H(\{a\})$, где $a \in A$. Условие же г′) соответствует амальгамируемости для биекций из S. Коррекция этого представления полугруппы $\mathcal{L}$ предпринятая в [2] связана лишь со строго минимальными идемпотентами полугруппы $\mathcal{L}$ и не приводит к нарушению требований $\vee$-неразложимости элементов вида $H(\{a\})$ и амальгамируемости биекций из S в откорректированном представлении. Тем самым для откорректи рованной пары $< H, S >$ оказываются выполненными условия теоремы 1 и, значит, найдется унар $\mathcal{A}$ такой, что пары $< \mathcal{L}, U >$ и $< \mathrm{Iso}\,\mathcal{A}, U_{\mathcal{A}} >$ изоморфны. Теорема доказана.

Так как для любого унара $\mathcal{A}$, любого натурального n матричная сте пень $\mathcal{A}^{[n]}$ остается унаром и унаром же будет η-редукт $\mathcal{A}^{[n]}(\eta)$ алгебры $\mathcal{A}^{[n]}$ по любому обратимому идемпотентному терму $\eta(x)$ сигнатуры $\sigma^{[n]}$, то из теоремы 2 вытекает

СЛЕДСТВИЕ 2. Конечная алгебра $\mathcal{A}$ условно рационально эквива лентна некоторому унару тогда и только тогда, когда пара $< \mathrm{Iso}\,\mathcal{A}, U_{\mathcal{A}} >$ удовлетворяет условиям а$'$) д$'$) и решетка идемпотентов полугруппы $\mathrm{Iso}\,\mathcal{A}$ дистрибутивна.

Заметим, что условия а$'$)-д$'$), в отличии от условий следствия 1, не конкретны, а абстрактны.

Рассмотрим теперь аналогичные вопросы для моноунаров. Пусть $< H, S >$ — пара, состоящая из алгебраической решетки H подмножеств множества A и совокупности S биекций между этими подмножествами. Рассмотрим дополнительно следующие условия:

α) для любого $a \in A$ либо существует $b \in A$ такой, что $H(\{b\}) = H(\{a\}) \setminus \{a\}$, либо $H(\{a\})$ — атом в решетке H;

β) для любого $a \in A$ совокупность $\{B \in H | B \subsetneq H(\{a\})\}$ линейно упорядочена по включению и либо конечна, либо имеет порядковый тип ω^*. Если $H(\{a\}$ атом в H, то $|H(\{A\})| < \chi_0$.

γ) если $|H(\{a\})| = |H(\{b\})|$ и $H(\{a\})$ атом в H, то $G_a = \{g \in S | d_g = r_g = H(\{a\})\}$ циклическая группа, $|G_a| = |H(\{a\})|$ и существует $h \in S$ такой, что $h(a) = b$.

Если $|H(\{a\})| = |H(\{b\})|$, $H(\{a\})$ не является атомом в H и $|\{B \in H | B \subsetneq H(\{a\})| = |\{B \in H | B \subsetneq H(\{b\})|$, то существует $g \in S$ такой, что $g(a) = b$.

ТЕОРЕМА 3. Для конкретно алгебраически допустимой пары $< H, S >$ на множестве A следующие условия эквивалентны:

а) для некоторого моноунара $\mathcal{A} =< A; f >$ пары $< H, S >$ и $< Sub\,\mathcal{A}, \mathrm{Iso}\,\mathcal{A} >$ совпадают,

б) пара $< H, S >$ такова, что решетка H дистрибутивна, для любого $a \in A$ элемент $H(\{a\})$ $\vee$-неразложим в решетке H, для S имеет место условие амальгамируемости и выполнены условия α)-γ).

Доказательство. Импликация а)$\to$б) очевидна. Докажем обратную импликацию б)$\to$а). Пусть пара $< H, S >$ удовлетворяет условию б). На множестве A определим функцию f следующим образом:

если $H(\{a\})$ не является атомом решетки H, то в силу условий α) и β) найдется единственный элемент $b \in A$ такой, что $H(\{b\}) = H(\{a\}) \setminus \{a\}$, и в этом случае, положим $f(a) = b$;

если $H(\{a\})$ атом решетки H, то, т.к. по условию β), множество $H(\{a\})$ конечно, пусть $H(\{a\}) = \{a_1, \ldots, a_n\}$. Для $i \leq n$ положим $f(a_i) = g(a_i)$, где g — некоторый фиксированный порождающий циклической группы $\{g \in S | d_g = r_g = H(\{a\})\}$.

Таким образом функция f определена на всем множестве A. Положим $\mathcal{A} =< A; f >$. В силу алгебраичности решетки подмножеств H для равенства $H = Sub\,\mathcal{A}$ достаточно показать, что для любых $a_1, \ldots, a_n \in A$ $H(\{a_1, \ldots, a_n\}) =< a_1, \ldots, a_n >$, где $< a_1, \ldots, a_n >$ подалгебра моноунара $\mathcal{A}$ порожденная элементами $a_1, \ldots, a_n$. В силу дистрибутивности H и $\vee$-неразложимости в решетке H подмножеств вида $H(\{a\})$, где $a \in A$, как и в доказательстве теоремы 1, замечается, что $H(\{a_1, \ldots, a_n\}) = \bigcup_{i=1}^{n} H(\{a_i\})$ и, тем самым, для равенства $H = Sub\,\mathcal{A}$ достаточно заметить, что $H(\{a\}) =< a >$ для любого $a \in A$. Последнее же непосредственно вытекает из построения функции f. Принцип глобализации и условия амальгамируемости для биекций из S сводит доказательство равенства $\{f | f$ — изоморфизм $\mathcal{V}$ на $\mathcal{L}\} = \{g \in S | d_g = \mathcal{V}, r_g = \mathcal{L}\}$, где $\mathcal{L}$ и $\mathcal{V}$ подалгебры алгебры $\mathcal{A}$, к доказательству равенств $\{f | f$ — изоморфизм $< a >$ на $< b >\} = \{g \in S | d_g =< a >, r_g =< b >\}$ для любых $a, b \in A$. Последнее же легко вытекает из принципа неподвижных точек и условия γ). Импликация б)$\to$а), а вместе с ней и теорема 3, доказаны.

СЛЕДСТВИЕ 3. Конечная алгебра $\mathcal{A}$ условно рационально эквивалентна некоторому моноунару тогда и только тогда, когда решетка $Sub\,\mathcal{A}$ дистрибутивна, для любого $a \in A$ подалгебра $< a >$ $\vee$-неразложима в решетке $Sub\,\mathcal{A}$, полугруппа $\mathrm{Iso}\,\mathcal{A}$ обладает свойством амальгамируемости и $< Sub\,\mathcal{A}, \mathrm{Iso}\,\mathcal{A} >$ удовлетворяет условиям $\alpha)$–$\gamma)$.

Укажем теперь условия при которых абстрактно алгебраически допустимая пара $< \mathcal{L}, U >$ изоморфна паре $< \mathrm{Iso}\,\mathcal{A}, U_{\mathcal{A}} >$ для некоторого моноунара. Рассмотрим следующие условия на инверсную полугруппу $\mathcal{L}$ и замкнутое подмножество U ее строго минимальных идемпотентов:

$\alpha')$ для любого $g \in \mathcal{L}$, удовлетворяющего условию (*) из условия д'), найдется $u \in \mathcal{L}$ удовлетворяющий условию (*) такой, что $\mathcal{L}uu^{-1} = \mathcal{L}gg^{-1} \setminus \{hgg^{-1}\}$, где h — элемент из условия (*);

$\beta')$ для любого $g \in \mathcal{L}$ удовлетворяющего условию (*) совокупность идемпотентов полугруппы $\mathcal{L}$ не превышающих идемпотент gg^{-1} линейно упорядочена и либо конечна, либо имеет порядковый тип ω^*. Если e атом в решетке идемпотентов полугруппы $\mathcal{L}$, то $|\mathcal{L}e| < \chi_0$;

$\gamma')$ если идемпотент $e \in \mathcal{L}$ является атомом в решетке идемпотентов полугруппы $\mathcal{L}$, то $G_e = \{g \in \mathcal{L} | eg = ge = g\}$ — циклическая группа, $|G_e| = |\mathcal{L}e|$ и для любого атома e_1 в решетке идемпотентов полугруппы $\mathcal{L}$ такого, что $|\mathcal{L}e_1| = |\mathcal{L}e|$ существует $h \in \mathcal{L}$ такой, что $eh = e_1$. Если $g_1, g_2 \in \mathcal{L}$ и удовлетворяют условию (*), элементы $h_1, h_2 \in \mathcal{L}$ играют роль элемента h из условия (*) для g_1 и g_2 соответственно, $|\mathcal{L}g_1g_1^{-1}| = |\mathcal{L}g_2g_2^{-1}|$ и числа идемпотентов меньших чем $g_1g_1^{-1}$ и $g_2g_2^{-1}$ совпадают, то существует $g \in \mathcal{L}$ такой, что $h_1g_1g_1^{-1}g = h_2g_2g_2^{-1}$.

Очевидно, что условия $\alpha')$–$\gamma')$ для пары $< \mathcal{L}, U >$ соответствуют условиям $\alpha)$–$\gamma)$ в представлении Вагнера-Престона. Кроме того, в силу равенства $|G_e| = |\mathcal{L}e|$ для атомов e решетки идемпотентов полугруппы $\mathcal{L}$, входящего в условие $\gamma')$, U будет состоять в точности из всех строго минимальных идемпотентов полугруппы $\mathcal{L}$. Тем самым, корректировка представления Вагнера-Престона в данном случае не потребуется. В силу изоморфизма пары $< \mathcal{L}, U >$ и пары $< \{\mathcal{L}gg^{-1} \setminus \{0\} | g \in \mathcal{L}\}, \{\varphi_g | g \in \mathcal{L}\} >$ по теореме 3 получаем следующее утверждение.

ТЕОРЕМА 4. Для любой алгебраически допустимой пары $< \mathcal{L}, U >$ следующие условия эквивалентны:

а) для некоторого моноунара $\mathcal{A}$ пары $< \mathcal{L}, U >$ и $< \mathrm{Iso}\,\mathcal{A}, U_{\mathcal{A}} >$ изоморфны,

б) пара $< \mathcal{L}, U >$ такова, что идемпотенты инверсной полугруппы $\mathcal{L}$ образуют дистрибутивную решетку и $\mathcal{L}$ удовлетворяет условиям г'), д'), $\alpha')$–$\gamma')$.

СЛЕДСТВИЕ 5. Для любой конечной алгебры $\mathcal{A}$ следующие условия эквивалентны:

а) алгебра $\mathcal{A}$ схожа с некоторым моноунаром;

б) пара $< \mathrm{Iso}\,\mathcal{A}, U_{\mathcal{A}} >$ удовлетворяет условию б) теоремы 4.

В заключение опишем алгебры условно рационально эквивалентные конечным полям рассматриваемым в сигнатуре $< +, \cdot, 0, 1 >$. Мощность любого конечного поля F равна p^n для некоторого простого числа p и натурального n. Как хорошо известно, см. к примеру, [8], подалгебрами конечных полей являются подполя и любой изоморфизм между подполями поля F продолжим до автоморфизма поля F, при этом простое подполе поля F — суть неподвижные точки всех автоморфизмов поля F. Группа автоморфизмов $Aut\,F$ поля Галуа $F = GF(p^n)$ — циклическая порядка n и существует взаимно однозначное соответствие (соответствие Галуа) между подгруппами группы $Aut\,F$ и подполями поля F (соответствующее подполе является совокупностью неподвижных точек соответствующей подгруппы). При этом, если H подгруппа группы $Aut\,F$ и $|h| = m$, то число неподвижных точек для H равно $p^{\frac{n}{m}}$. Таким образом, если $\mathcal{A} =< A; \sigma >$ универсальная алгебра мощности p^n для некоторого простого p и натурального n, группа автоморфизмов алгебры $\mathcal{A}$ циклическая порядка n, для любого m (делителя n) неподвижные точки автоморфизмов алгебры $\mathcal{A}$ порядка m образуют подалгебру алгебры $\mathcal{A}$ мощности $p^{\frac{n}{m}}$ и других подалгебр алгебры $\mathcal{A}$ нет, кроме того любой автоморфизм подалгебры алгебры $\mathcal{A}$ продолжим до автоморфизма самой $\mathcal{A}$, то, очевидно, существует взаимно однозначное отображение π основного множества A алгебры $\mathcal{A}$ на основное множество поля $GF(p^n)$ такое, что $\pi(Sub\,\mathcal{A}) = Sub\,GF(p^n)$ и π сопрягает отображения из $\mathrm{Iso}\,\mathcal{A}$ с отображениями из $\mathrm{Iso}\,GF(p^n)$. Тем самым имеет место

ТЕОРЕМА 5. Для конкретно алгебраически допустимой пары $< H, S >$ на множестве A следующие условия эквивалентности:

а) для некоторого поля $F =< A; +, \cdot, 0, 1 >$ пары $< H, S >$ и $< Sub\,F, \mathrm{Iso}\,F >$ совпадают, б) $|A| = p^n$ для некоторого простого p и натурального n, любое отображение из S продолжимо до перестановки множества A входящей в S, перестановки A входящие в S образуют циклическую группу G порядка n, подмножества входящие в H суть подмножества неподвижных точек для подгрупп группы G и при этом если G_1 — подгруппа группы G состоящая из элементов порядка m (m — делитель n), то мощность множества неподвижных точек для G_1, равна $p^{\frac{n}{m}}$.

СЛЕДСТВИЕ 6. Конечная универсальная алгебра $\mathcal{A}$ схожа с полем $GF(p^n)$ тогда и только тогда, когда $< \mathrm{Iso}\,\mathcal{A}, U_{\mathcal{A}} > \cong < \mathcal{L}, \emptyset >$, где $\mathcal{L}$ инверсная полугруппа являющаяся связкой фактор-групп группы Z_n относительно решетки идемпотентов двойственной к решетке делителей числа n, где образующими гомоморфизмами между этими фактор-группами являются канонические гомоморфизмы.

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COVERS OF ALGEBRAIC VARIETIES[1]

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1 Covers of algebraic varieties.

The investigation of covers of algebraic varieties has a long history. I'll give you a short sketch of it.

Let us remember some terminology. Let K be a field of characteristic p. Let V be an algebraic K-variety (i.e. V is ccdefined in some projective or an affine space by polynomials with coefficients in the field K). By *a cover* of V is called a dominant regular K-rational finite map f of some K-variety U into V, $f : U \to V$. Here K-rational and regular means that f can be defined by K-polynomials and dominant means that image of U in V is dense in Zarisky topology and finite means that preimage of any point of V is a finite set of points of U.

Let $K(V)$ and $K(U)$ be the fields of K-rational functions of these varieties. The cover f corresponds to the embedding of the fields $f^* : K(V) \to K(U)$, i.e. to some extension of the field $K(V)$. The degree of this extension is called *degree of f*. If this extension is a separable one then the cover f is called a *separable cover*. From now on we shall consider only separable covers. Degree of the cover f let us denote by n.

Now let a field L be a normal closure of the extension f^*. This is a minimal Galois extension of $K(V)$ which contains $K(U)$. Let G be a galois group of the extension L/K. In some sense this group is a measure of complexity of the cover f. The cover f is called *unsolvable, soluble, abelian or cyclic cocver* if the group G is an unsoluble, soluble, abelian or cyclic group correspondingly.

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Let us assume that variety V is a normal variety (it means that for any point of V a local ring is an integrally closed ring). In this case almost any point of V has exactly n preimages. In other points of V the number of preimages is less then n. These points are called *branched points* of f. The set of branched points is called *branch loci* of f.

From general ramification theory (see [1]) it follows that branch loci is a proper subvariety of V. If variety V is a regular variety and U a normal variety then well-known theorem of Zarisky-Nagata about the purity of branch loci asserts that the branch loci of f is a finite union of subvarieties of codimension one (loc.cit.). So usually one impose just this condition on U and V.

From now on I'll consider V to be a regular variety and U be a normal variety. General problem in the investigation of cover f is to find a connection between a character of singular points of a branch loci of f and a galois group of the cover.

Let us give some example. In this example I restrict ourself to affine varieties only.

Example. Let K be a field. Let p does not equal to 5. Let U be a surface in an affine 3-dimensional space defined by the equation: $x^5 - xz - y = 0$. Let V be an affine plane of coordinates X and Y and f be a projection of U into V. From galois theory it follows that galois group of f is the alternation group on five symbols. This group is unsoluble. Moreover it is known that in the origin the local Galois group of this cover is unsoluble (this statement belongs to A.Hovansky [2]). In this example the branch loci C is a plane curve. In the origin this curve has a singularity - a pinch singularity (i.e., in the origin two tangent lines to this curve coincide).

An other kind of singularity of a plane curve is a node singularity, i.e. in the singular point two tangent lines to this curve are distinct lines.

Let now K be a field of complex numbers, let V and C be as in the above example. Suppose that C has only nodal singular points. This time it is known that if $f : U \to V$ is a cover and the branch loci of f are contained in the curve C then the cover f is an abelian cover. This follows from investigation of Zarisky problem by S.Abhyankar, W.Fulton e.t.c.. In the section 3 I'll give some remarks about this problem.

2 Invariants of branched covers.

My interest to the problem is concerned with the case of a nonzero characteristic of the field of definition F. Denote by the prime p a characteristic exponent of the field F ($p = 1$ if characteristic of F is equal to zero). Let us define some characteristics of covers. From now on let U and V be projective algebraic varieties, V be a projective and U be a normal variety. Let us denote by $K = F(V)$ and by $L = F(U)$ the fields of rational functions of these varieties. As before a cover $f : U \to V$ defines an inclusion $K \subset L$. So one has an extension L/K.

Projective variety is a complete variety. So irreducible subvarieties of codimension one corresponds to discrete valuations of the field of rational functions (see [3]). From ramification theory it follows that irreducible components of the branch loci correspond to those valuations of the field K that have ramified extensions on the field L.

So we come to the notion of ramification index. Let w be some discrete valuation of the field K. If the extension L/K is a galois extension then all continuation of this valuation to the field L are conjugated by automorphisms from galois group. From now on let us assume that L/K is a galois extension and the cover f is a galois cover. This restriction is necessary in the statements of the author too.

Let W be a subvariety of V of codimension one. Let it corresponds to the valuation w of K. Let W' be a preimage of W in U and let W'' be some irreducible component of W'. Let v be a valuation of L which corresponds to W''. This valuation is an extension of the valuation w. So the value group $w(K)$ is a subgroup of $v(L)$, it has a finite index. This index is called *reduced ramification index* of the cover f on the variety W, $e_r^W(f)$.

Residue fields of the valuations v and w correspond to the fields of rational functions of the varieties W and W'' correspondingly. Let us denote it by k and l. Then one can consider k to be a subfield of l. This inclusion corresponds to the restriction of the cover f on the variety W'', This restriction is a cover but it need not be a separable one. Let us denote by $d^W(f)$ a degree of the extension l/k, and by $d_u^W(f)$ unseparable degree. The number d_u^W is some degree of p.

DEFINITION 1. By *ramification index of the cover f on the variety W* is called a natural $e_W(f)$ defined by formula $e_W(f) = e_r^W(f)d_u^W(f)$. (see [4]). It is easy to see that W is an irreducible component of the branch loci iff $e_W(f) \neq 1$. If $e_W(f) = 1$ then f is called by *unbranched cover* on W. This case the extension l/k is a separable and valuation w of the field K is unramified in the field L.

DEFINITION 2. The cover f is called *unbranched cover* if it is unbranched cover for any proper subvariety of codimension one.

A grater class of covers consists of tamely branched covers.

DEFINITION 3. The cover f is called *tamely branched cover* if for any proper subvariety W of codimension one the ramification index f on W is relatively prime to p, $(p, e_W(f)) = 1$.

If the field F has zero characteristic then any cover is a tamely branched cover by definition. In nonzero characteristic p there are many non tamely ramified covers. The definition 3 means that for any subvariety W of codimension one the extension of residue fields l/k is a separable one and reduced ramification index is relatively prime to p.

More greater class of covers consists of separably branched covers.

DEFINITION 4. The cover f is called *separably branched cover* if for any proper subvariety W of codimension one the extension of residue fields l/k is a separable extension.

The author has proved the assertion.

THEOREM 1. Let V be a projective plane defined over a field F. Let p be a characteristic exponent of K. Let C be some curve in V which has only nodal singularity. Then if $f : U \to V$ is a galois cover which is separably branched along the curve C then this cover f is a soluble one.

3 Fundamental groups, tame fundamental groups. Counterexamples.

In this section I point out a place of before-mentioned theorem 1 in theory of algebraic covers. I shall use the notation of this theorem.

Let us remember that by a quasi-projective variety is called a supplement with a proper subvariety of a projective variety. For instance, an affine space is a quasi-projective variety. Let us note that if a cover f is branched only along some proper subvariety W then one can consider f as an unbranched cover of a quasi-projective variety $U' = U\backslash W$.

A family of all unbranched covers of a given quasi-projective variety forms an algebraic fundamental group of this variety. This notion was introduced by S.S.Abhyankar in [5]. Let us give a precise definition.

Let M' be a quasi-projective variety defined over F. Let $K = F(M')$ be a field of rational functions of M'. Let Ω be an algebraic closure of this field. Covers of M' by normal varieties are in one-to-one correspondence to finite extensions of the field F in Ω.

For any finite extension L/K there exist a normalization of the variety M' in L. This is a cover $f_L : N \to M'$, where $F(N) = L$ and N is a normal variety. This cover is a unique in a proper sense.

Let us denote by Ω_{ur} a family of such finite galois extensions L/K in Ω that f_L is an unbranched cover of M' (i.e. f_L branched only along boundary W of M'). This family forms a direct system of subfields of Ω. So galois groups of these fields over K form an inverse system of finite groups. Inverse limit of this family determines a profinite group. This group is called *algebraic fundamental group*, it is denoted by $\pi^{alg}(M')$ or $\pi(M')$.

This group was investigated in connection with Zarisky problem. This problem is the following. Let F be the field of complex numbers, let V be a regular quasi-projective variety over F of dimension two and its boundary C is a curve with nodal singularities only. One can consider V as an analytic manifold over the field of real numbers of the dimension 4. This manifold is connected and one can define a topological fundamental group of this manifold $\pi^{top}(V)$. Problem of Zarisky asks: does it true that the group $\pi^{top}(V)$ is an abelian group ?

The connection between the group $\pi^{top}(V)$ and the group $\pi^{alg}(V)$ is defined by Generalized Rieman Existence Theorem of Grauert and Remmert [6]. From this theorem follows that the group π^{alg} is a profinite completion of the group π^{top}. So the group $\pi^{top}(V)$ is an abelian group iff the group $\pi^{alg}(V)$

is an abelian. It follows from the example 1 that the conditions on singular curve is a necessary condition for the problem.

Historically first algebraic approach to the problem was concerned with algebraic fundamental group [5]. Now there are many topological solutions of this problem for the case of the complex numbers of course.

Let us consider a case of a field F of a nonzero characteristic p. This case the construction of the group $\pi^{alg}(V)$ is rather difficult. S.Abhyankar pointed out the examples of unsolvable galois covers of the projective plane which branched only along one line [7]. So this case one restrict ourself to investigation of tame fundamental group π_t.

Let us give a precise definition. Denote by Ω_{tr} a subsystem of Ω_{ur} which contains only tamely branched covers. This is a direct system and one has an inverse system of profinite groups. Inverse limit of this system is called tame fundamental group. By construction this group is a factor of the group π^{alg}. If F is a field of zero characteristic then $\pi^{alg}= \pi_t$. But if $p \neq 0$ the group π_t is a nontrivial factor of the group π.

In full generality Zarisky problem was solved by W.Fulton in 1980 [8]. He has proved:

THEOREM. Let M be a projective plane and C a plane curve in M whose singularities are only nodes. Then the group $\pi_t(M \setminus C)$ is an abelian group.

The equivalent form of this statement is the following:

COROLLARY. In the conditions of the theorem let N be a normal variety and $f : N \to M$ a tamely branched cover of M branched only along C. Then this cover is an abelian cover.

So the above mentioned Theorem 1 of the author is a generalization of this statement in the case when f is only separably branched galois cover. I point out that in the statement of this theorem the condition "to be a galois cover" is necessary. Indeed one can deduce corresponding example from the example no.5 [7]. Besides, it follows that separably branched covers do not form direct system of fields. So one can't construct "separable fundamental group". The author gives more general theorem about covers of a projective n-space.

THEOREM 2. Let M be a projective n-space over the field of F of char-

acteristic p, $p \neq 0$. Let C be a subvariety of M of codimension one (it may be a reducible variety). Let an intersection of C by a generic surface of M is a curve with nodal singularities only. Then if $f : N \to M$ is a galois cover branched separably along C only then f is a soluble cover.

4 Fields with discrete valuations.

Some words about the proof of the theorem 1. In section 2 I pointed out on the importance of discrete valuations. Key point in the work of S.Abhyankar [5] is the assertion known as Abhyankar's lemma.

ABHYANKAR LEMMA. Let K be a field with a discrete valuation w. Let this field has all roots from the unit.

Let L/K be a galois extension and let w be tamely ramified in L. Then in some greater field there exist a cyclic galois extension K' of the field K that the extension LK'/K' is unramified for any continuation of w into LK'.

If the valuation w is not tamely ramified in L then it is impossible to prove similar result. It follows from a rather horrible structure of unbranched covers of affine line (see [9]). But the author has proved the following assertion ([10]).

THEOREM. Let L/K be a finite separable extension of a discrete valued field K of a nonzero characteristic p. Let this field has all roots from the unit and let residue field is a functional field.

Then in the separable algebraic closure of the field K there is a solvable finite extension K' of K that the extension $L \cdot K'/K'$ is tame unramified for any continuation of the valuation of K on the field $L \cdot K'$.

The possibility to use this theorem instead of Abhyankar's lemma follows from Fulton's theorem about connectedness [8].

5 Semialgebraic sets in local fields.

It is well known that a list of local fields of characteristic zero comprises the following: the field of complex numbers $\mathbf{C}$, the field of real numbers $\mathbf{R}$, the field of p-adic numbers $\mathbf{Q}_p$, where p is any prime, and also all finite extensions of these fields. Let K be a local field of characteristic zero. A subset of the direct power K^n is called a *semialgebraic set* if it is a Boolean combination of subsets of the form $\{\bar{x} \in K^n | (\exists y \in K) y^m = f(\bar{x})\}$ for different polynomials $f(\bar{x})$ in n variables with coefficients from K, $f \in K[\bar{x}]$, and for natural $m \in \mathbf{N}$ (see [4]). For the case $K = \mathbf{C}$, this definition coincides with that of a constructible set.

The importance of this notion to the model theory of local fields is explained by the following. If we enrich the signature of the theory of fields by unary predicates which connote the sets of mth powers: $\Sigma' = (+, \cdot, \uparrow m \mid m \in \mathbf{N})$, then semialgebraic sets in K^n are exactly those which are defined by quantifier-free formulas in the signature Σ'.

In the paper [11] the author had used this notion and had gotten a new proof of Tarsky-Seidenberg-Macintyre theorem on elimination of quantifiers. The author gave a new scheme for investigation of definable sets in local fields. Initial point of this research is a semistable reduction theorem from algebraic geometry (see [12]). Some pure version of this theorem was formulated in the last section of the paper. Now the assertion of semistable reduction theorem is known for the case of zero characteristic only.

The above mentioned theorem is a crutial step in the proof of semistable reduction theorem for fields of nonzero characteristic.

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ОПИСАНИЕ ГРУПП ЕДИНИЦ НЕПРИВОДИМЫХ МАТРИЧНЫХ КОЛЕЦ[1]

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Проблема описания мультипликативной структуры различных колец сформулирована Л.Фуксом в известных монографиях ([1],[2]). В предлагаемой работе рассматривается частный случай, а именно конечно порожденные матричные кольца, неприводимые над полем рациональных чисел. Выбор именно неприводимых колец диктуется простым соображением. Известно, что произвольное матричное кольцо над полем рациональных чисел приводится к клеточно-треугольному виду, в котором диагональные клетки либо абсолютно неприводимы, либо неприводимы над Q. Поскольку в работе автора [3] доказывается эффективность клеточно-треугольного представления, то имеет смысл решение поставленной задачи прежде всего для отдельной клетки. Пусть O_i - такая клетка. Обозначим через $[O_i]_Q$ линейную оболочку кольца O_i над полем рациональных чисел. Могут представиться три случая:

1) $[O_i]_Q = Q_{n_i}$;

2) $[O_i]_Q \cong F_k$, F - поле;

3) $[O_i]_Q \cong T_k$, T - тело.

Первый случай относится к абсолютно неприводимым кольцам, второй и третий - к кольцам, неприводимым над Q. Для первых двух случаев задача описания группы единиц $U(O_i)$ кольца O_i на языке порождающих элементов была решена автором в [4] и [5]. Таким образом, решение задачи для третьего случая делает возможным описание мультипликативных групп произвольных конечно порожденных матричных колец, неприводимых над Q, что, в свою очередь, позволит описать мультипликативную

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структуру произвольного конечно порожденного матричного кольца над полем рациональных чисел.

1 Строение неприводимых над Q колец

Пусть $O = rg(x_1, ..., x_r) \subset Q_n$ - неприводимо над Q, при этом $[Q]_Q \cong H_k$, где $H \subset Q_l$ - тело размерности $l = \frac{n}{k}$ (см.[6])

Аналогично [7] доказывается

ТЕОРЕМА 1. Если конечно порожденное кольцо $O \subset Q_n$ абсолютно неприводимо или неприводимо над Q , то эффективно находятся такие элементы $g_1, ..., g_t \in O$ и такое множество простых чисел $\pi =< p_1, ..., p_s >$, что

$$O = \{g_1, ..., g_t\}_{Z_\pi},$$

где $Z_\pi = Z[\frac{1}{p_1 \cdots p_s}]$.

Набор элементов $B =< g_1, ..., g_t >$ назовем базой кольца O , а кольцо Z_π - кольцом коэффициентов кольца O .

Обозначим $H_\pi = H \cap Z_\pi$. Будем считать, что $[O]_Q = H_k$. Из Т.1 легко следует, что найдется ненулевое натуральное число m , такое, что $(m, p_1 \cdots p_s) = 1$ и для которого справедливо включение:

$$mH_{\pi^\perp} \subset O$$

В кольце H_π породим идеал $I = (me_l)$, где e_l - единичная матрица из Q_l. Определим гомоморфизмы:

$$\gamma_m : H_\pi \longrightarrow H_\pi / I = \overline{H_\pi}$$

$$\psi_m : H_{\pi^\perp} \longrightarrow \overline{H_{\pi^\perp}}$$

$$\varphi_m : GL_k(H_\pi) \longrightarrow GL_k(\overline{H_\pi})$$

$$U_0(O) = U(O) \cap ker\varphi_m$$

Порождающие $U(O)$ можно находить в два этапа:

1) порождающие группы $U_0(O)$;

2) порождающие группы $\overline{U(O)} = U(O)/U_0(O)$.

Строение группы $ker\varphi_m$ описано в [10] для дедекиндовых колец. H_π, вообще говоря, не удовлетворяет этому условию. Остановимся на этом подробнее. В силу неприводимости O по лемме Шура $\Gamma = Hom_O(Q^n, Q^n)$ является телом. Тогда по альтернативе Титса группа $\Gamma^* = \Gamma\setminus\{0\}$ либо почти разрешима, либо содержит F_2 - свободную группу ранга 2.

Можно показать, что если Γ^* - почти разрешима, то H - поле и H_π - дедекиндово кольцо. Т.е. это в точности случай 2), рассмотренный в [5].

2 Порождающие $U(H_\pi)$

Итак, пусть теперь H - не коммутативно. Известно (см.[8]), что в этом случае $[H:Q]=d^2$ и в H существуют два подполя $F=\{1,f,...,f^{d-1}\}_Q$ и $T=\{1,t,...,t^{d-1}\}_Q$ такие, что

1) $H=\{f^it^j|i,j=0,...,d-1\}_Q$,

2) $\exists\xi\in H[f=\xi^{-1}t\xi]$

Отсюда следует, что для любого $h\in H$ справедливо представление $h = ab$, где $a\in F, b\in T$. Действительно, пусть $h=\sum_{i,j=0}^{d-1}\alpha_{ij}f^it^j$. Чтобы найти $a=\sum_{i=0}^{d-1}\beta_if^i$ и $b=\sum_{j=0}^{d-1}\gamma_jt^j$ достаточно заметить, что $ab=\sum_{i,j=0}^{d-1}\beta_i\gamma_jf^it^j$, т.е. должны выполняться равенства $\beta_i\gamma_j=\alpha_{ij}, i,j=0,...,d-1$. Понятно, что для данных $\alpha_{ij}\in Q$ всегда можно найти подходящие β_i и γ_j. Следовательно, $H_\pi = F_\pi T_\pi$. Поскольку F и T - алгебраические расширения Q, то для максимальных порядков в них существует теория дивизоров (см.[9]), с помощью которой можно найти порождающие групп $U(F_\pi)$ и $U(T_\pi)$ и, тем самым, порождающие группы $U(H_\pi)=U(F_\pi)U(T_\pi)$.

3 Порождающие $ker\gamma_m^*$

Обозначим сужение γ_m^* на $U(F_\pi)$ и $U(T_\pi)$ соответственно σ_m^* и τ_m^*. В [5] доказывается эффективность нахождения порождающих для $ker\sigma_m^*$

и $ker\tau_m^*$. Ясно, что эти порождающие являются частью множества порождающих $ker\gamma_m^*$. Также понятно, что ими множество порождающих $ker\gamma_m^*$ не исчерпывается, т.к. может быть, что $(e_l+a)\in U(F_\pi)\setminus ker\sigma_m^*$, $(e_l+b)\in U(T_\pi)\setminus ker\tau_m^*$ и при этом $(e_l+a)(e_l+b)\in ker\gamma_m^*$. Находить такие элементы можно конечным перебором, если заметить следующее. Т.к. $\gamma_m^*(U(H_\pi))$ конечна, то всякий элемент $\gamma_m^*(e_l+a)$ имеет конечный порядок, т.е. существует такое k, что $(\gamma_m^*(e_l+a))^k = \bar{e}_l$ или $\gamma_m^*((e_l+a)^k)=\bar{e}_l$, т.е. $(e_l+a)^k\in ker\gamma_m^*$. Поэтому для нахождения порождающих $ker\gamma_m^*$ достаточно перебрать конечное число слов от порождающих $U(H_\pi)$. Таким образом справедлива

ЛЕММА 1. Группа $ker\gamma_m^*$ конечно порождена, её порождающие находятся эффективно.

Обозначим $ker\gamma_m^* = gp(\tau_1,...,\tau_q)$.

4 Условие стабильности ранга

Пусть A - произвольное кольцо с 1, $\wp$ - двусторонний идеал A. Элемент $\alpha=(a_1,...,a_n)\in A^n$ называется унимодулярным в A^n, если существует гомоморфизм $f:A^n\longrightarrow A$, такой, что $f(\alpha)=1$. Это, очевидно, эквивалентно тому, что $\sum Aa_i = A$. Если, кроме того, $\alpha\equiv(1,0,...,0) \pmod\wp$, то говорят, что элемент α $\wp$-унимодулярен. Обозначим множество унимодулярных элементов $Um_n(A)$, а множество $\wp$-унимоулярных элементов $Um_n(A,\wp)$.

Определение. Условие $SR_n(A,\wp)$ (стабильности ранга).

Если $m\ge n$ и если $\alpha=(a_1,...,a_m)\in A^m$ есть $\wp$-унимодулярный элемент, то существуют элементы $a_i'=a_i+b_ia_m$, где $b_i\in\wp (1\le i<m)$ такие, что элемент $(a_1',...,a_{m-1}')\in A^{m-1}$ унимодулярен. (В [11] показанно, что достаточно потребовать выполнение условия при $m=n$, тогда оно будет выполненно и при $m\ge n$). Если $\wp=A$, то условие обозначим $SR_n(A)$.

Для нашего случая $A=H_\pi$, $\wp=I$.

ЛЕММА 2. Условие $SR_3(H_\pi, I)$ выполняется. Прежде всего заметим, что $SR_n(A)\Longrightarrow SR_n(A,\wp)$ для всех $\wp$ (см.[13]). Поэтому достаточно доказать выполнение $SR_3(H_\pi)$.

Для этого рассмотрим подробнее линейное строение H_π.

1) Т.к. H_π - область целостности ,то любая матрица из H_π опреде-
ляется одной строкой (или столбцом), например, первой. Поэтому мо-
жем считать, что аддитивный базис H_π состоит из первых строк матриц
$\{e_l, f, ..., f^i t^j, ..., f^{d-1} t^{d-1}\}$. Переработаем этот базис так, что во всех ма-
трицах, кроме первой, элемент, стоящий на месте (11), равняется нулю.
Пусть

$$H_\pi = \{e_l, h_1, ..., h_{l-1}\} z_\pi,$$

где

$$h_i = \begin{pmatrix} 0 & h^i_{12} & \cdots & \cdots & h^i_{1l} \\ & & & & \\ & & * & & \\ & & & & \end{pmatrix}$$

Обозначим Δ_{H_π} определитель, составленный из первых строк элемен-
тов базиса. Пусть $\Delta_{H_\pi} = k$. (Для простоты рассуждений без ограничения
общности можно считать множители из Z_π равными 1).

2) Всякий идеал из H_π определяется определителем, составленным из
первых строк элементов его аддитивного базиса. Понятно, что каждый
идеал обладает базисом с минимальным определителем, причем для лю-
бого собственного идеала этот определитель равен $k'k$, где $k' > 1$.

3) Пусть $a \in H_\pi \setminus U(H_\pi)$ и пусть $|a| = k' \neq 1$. Тогда идеал
$H_\pi a = \{a, h_1 a, ..., h_{l-1} a\} z_\pi$. Понятно, что $\Delta_{H_\pi a} = |a| k = k'k$. Из фор-
мул Крамера тогда следует, что $k' e_l \in H_\pi a$, причем k' - минимальное
ненулевое натуральное число с таким свойством. Рассмотрим идеал I с
базисом:

$$I = \{k' e_l, h_2, ..., h_{l-1}\} z_\pi$$

Т.к. $\Delta_I = k'k$, то $I = H_\pi$. Отсюда следует, что

$$a = \alpha k' e_l + \sum_{i=2}^{l-1} \alpha_i h_i \qquad (1)$$

Теперь переходим к доказательству условия $SR_3(H_\pi, I)$. Пусть
$(a_1, a_2, a_3) \in Um_3(H_\pi)$. Это означает, что $H_\pi a_1 + H_\pi a_2 + H_\pi a_3 = H_\pi$.
Если $|a_i| = k_i (i = 1,2,3)$ и $a_i = \alpha_i k_i e_l + \sum_{j=2}^{l-1} \alpha^i_j h_j$, то, понятно, что
$(\alpha_1 k_1, \alpha_2 k_2, \alpha_3 k_3) = 1$. Если $(\alpha_1 k_1, \alpha_3 k_3) = d_1$, $(\alpha_2 k_2, \alpha_3 k_3) = d_2$, то в про-
грессиях $\alpha_1 k_1 + t\alpha_3 k_3$ и $\alpha_2 k_2 + q\alpha_3 k_3$ по теореме Дирихле содержится беско-
нечно много чисел вида $p_1 d_1$ и $p_2 d_2$, где p_1, p_2 - простые. Стало быть, су-
ществуют такие $u, v \in Z$, что $(\alpha_1 k_1 + u\alpha_3 k_3, \alpha_1 k_1 + v\alpha_3 k_3) = 1$. А тогда эле-
мент $(a_1 + ua_3, a_2 + va_3) \in Um_2(H_\pi)$, т.к. $a_1 + ua_3 = (\alpha_1 k_1 + u\alpha_1 k_1) e_l + \sum_{i=2}^{l-1} \beta_i h_i$,
$a_2 + va_3 = (\alpha_2 k_2 + v\alpha_1 k_1) e_l + \sum_{i=2}^{l-1} \gamma_i h_i$ и $H_\pi(a_1 + ua_3) + H_\pi(a_2 + va_3) = H_\pi$.
Тем самым лемма доказанна.

5 Порождающие группы $U(O)$

ЛЕММА 3. Порождающие группы $GL_k(H_\pi, I) = ker\varphi_m$ находятся эффек-
тивно.

Для доказательства леммы сошлемся на следствие (4.5)(см.[13]), ко-
торое утверждает, что если A - конечномерная R-алгебра, где R -
коммутативное кольцо такое, что $max(R)$ - нетерово пространство,
являющееся объединением конечного числа подпространств размерности
Крулля $\leq d$, то утверждение теоремы (4.1) справедливы для кольца A
при $n = d + 2$.

Поскольку H_π - конечномерная Z_π -алгебра , а $dim(Z_\pi) = 0$, то утвер-
ждение теоремы (4.1) справедливо при $n = 2$. Тогда с учетом леммы 3
для $k \geq 3$ получаем

$$GL_k(H_\pi, I) = E_k(H_\pi, I) GL_1(H_\pi, I) \qquad (2)$$

Осталось заметить, что $GL_1(H_\pi, I) = ker\gamma^*_m$ и по лемме 2 её порожда-
ющие находятся эффективно.Тем самым лемма доказанна.

Как уже отмечалось выше , для нахождения образующих $U(O)$ нужно
прежде всего найти образующие $U_0(O) = U(O) \cap ker\varphi_m$.

Известно (см.[6]), что любая матрица $g \in GL_k(H)$ может быть представлена в виде $g = sd(\mu)$, где $d(\mu) = diag(1,...,1,\mu)$, $s \in SL_k(H)$. Нетрудно заметить, что и над H_π справедливо представление $g = sd(\mu)$, где $s \in SL_k(H_\pi)$, $\mu \in H_\pi$.

Если $g = sd(\mu) \in U_0(O) = U(O) \cap ker\varphi_m$, то, аналогично [5], $s \in SL_k(H_\pi) \cap O$, $d(\mu) \in U(O)$, $\mu \in ker\gamma_m^*$. Теперь понятно , что для нахождения образующих $U_0(O)$ нужно уметь находить образующие $SL_k(H_\pi) \cap ker\varphi_m$. Последнее легко следует из леммы 2.

Обозначим $\overline{O} = \{\bar{g}_1, ..., \bar{g}_t\}_{\bar{Z}_\pi}$, где $\bar{g}_i = \psi_m(g_i)$, $\overline{Z_\pi} = Z_\pi/(gn)$. Поскольку $\overline{U(O)} \subset U(\overline{O})$, а последняя конечна и её элементы находятся эффективно, то удобно находить порождающие $\overline{U(O)}$, перебирая элементы $U(\overline{O})$. Ясно , что $\bar{u} \in \overline{U(O)} \iff \psi_m^{-1}(\bar{u}) \cap U(O) \neq \emptyset$.

ЛЕММА 4. Пусть $\bar{u} \in U(\overline{O})$, $g \in \psi_m^{-1}(\bar{u})$, $g = sd(\mu)$, тогда

$$\exists h \in H_\pi, [g + mh \in U(O)] \iff \exists \mu' \in U(H_\pi), s' \in SL_k(H_\pi)[g + mh =$$

$$= s'd(\mu'), \mu \equiv \mu' \pmod I, s' \equiv s \pmod I].$$

Доказательство дословно повторяет доказательство аналогичной леммы в [5].

ЛЕММА 5. Существует алгоритм, позволяющий находить порождающие $\overline{U(O)}$.

Доказательство с незначительными изменениями повторяет доказательство аналогичной леммы в [5].

Таким образом справедлива

ТЕОРЕМА 2. Пусть $O = rg(x_1, ..., x_r) \subset Q_n$ неприводимо над Q. Тогда задача нахождения порождающих группы единиц $U(O)$ алгоритмически разрешима.

В заключение заметим, что полученный результат вместе с предыдущими результатами автора позволяет описать группы единиц произвольных конечно порожденных матричных колец над полем рациональных чисел и тем самым решить частный случай проблемы Л.Фукса в классе таких колец.

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ON THE CONGRUENCE INTERSECTION PROPERTY

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1 Introduction

The weak congruence lattice was introduced in [7]. Necessary notions and corresponding properties listed in this Introduction are taken from [4,5,7].

An algebra $\mathcal{A} = (A, F)$ is said to have the **Congruence Intersection Property** (CIP) if

$$(\rho \cap \theta)_A = \rho_A \cap \theta_A,$$

for each $\rho \in Con\mathcal{B}$, $\theta \in Con\mathcal{C}$, where $\mathcal{B}$ and $\mathcal{C}$ are arbitrary subalgebras of $\mathcal{A}$, and $\rho_A = \bigcap(\sigma \in Con\mathcal{A} \mid \rho \subseteq \sigma)$.

It is obvious that $\mathcal{A}$ has the CIP if and only if the diagonal relation Δ of $\mathcal{A}$ is a distributive element in the lattice $Cw\mathcal{A}$ of weak congruences of $\mathcal{A}$, i.e., iff

$$\Delta \vee (\rho \wedge \theta) = (\Delta \vee \rho) \wedge (\Delta \vee \theta), \tag{1}$$

for any $\rho \in Con\mathcal{B}$, $\theta \in Con\mathcal{C}$, where $\mathcal{B}$ and $\mathcal{C}$ are arbitrary subalgebras of $\mathcal{A}$.

It is easy to see that the dual of (1) holds for every $\mathcal{A}$. Indeed,

$$\Delta \wedge (\rho \vee \theta) = (\Delta \wedge \rho) \vee (\Delta \wedge \theta), \tag{2}$$

for any $\rho \in Con\mathcal{B}$, $\theta \in Con\mathcal{C}$. In other words, Δ is always a codistributive element in $Cw\mathcal{A}$. The function $m_\Delta : \rho \mapsto \rho \wedge \Delta$ which maps each congruence on a subalgebra of $\mathcal{A}$ to its diagonal relation is a lattice homomorphism from $Cw\mathcal{A}$ to $Sub\mathcal{A}$. Blocks of the corresponding congruence relation on the lattice $Cw\mathcal{A}$ are congruence lattices of subalgebras of $\mathcal{A}$.

From the lattice-theoretic point of view, each notion and property of $Cw\mathcal{A}$ introduced by codistributivity of the diagonal relation, has its dual if $\mathcal{A}$ has the CIP (recall that $\mathcal{A}$ has the CIP iff Δ is a distributive element in $Cw\mathcal{A}$). Namely, there is a congruence relation on $Cw\mathcal{A}$ induced by the map $n_\Delta : \rho \mapsto \rho \vee \Delta$, which is dual to m_Δ. Blocks of that congruence are the dual analogon of congruence lattices of subalgebras of $\mathcal{A}$. It turns out that in algebraic terms, these blocks can be considered as *lattices of subalgebras associated to congruences of $\mathcal{A}$*. This situation is investigated in the sequel.

Recall that in the weak-congruence lattice, subalgebras are represented by diagonal relations. Hence, the lattice of subalgebras of $\mathcal{A}$ is, up to the isomorphism, the principal ideal $\Delta\downarrow$ in $Cw\mathcal{A}$, generated by the diagonal of $\mathcal{A}$.

2 Results

Let $\mathcal{A} = (A, F)$ be an algebra and for $\theta \in Con\mathcal{A}$ let

$$C_\theta := \{\rho \in Cw\mathcal{A} \mid \rho_A = \theta\}. \tag{3}$$

Proposition 1 *If an algebra $\mathcal{A}$ has the CIP then $Con\mathcal{A}$ is a retract in $Cw\mathcal{A}$ under $\rho \mapsto \rho \vee \Delta$, and for every $\theta \in Con\mathcal{A}$, $(C_\theta, \subseteq)$ is a lattice, a sublattice of $Cw\mathcal{A}$.*

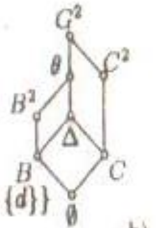
Proof. Obvious, since in the presence of the CIP, the map $\rho \mapsto \rho \vee \Delta$ is an endomorphism on $Cw\mathcal{A}$. □

Thus, the CIP is characterized by the lattices C_θ, $\theta \in Con\mathcal{A}$. However, for the converse, i.e., for an algebra $\mathcal{A}$ to have the CIP, it is not enough that the sets C_θ, $\theta \in Con\mathcal{A}$ are lattices. A simple example is a four-element groupoid G, given by its table in Fig. 1 a). For each of its congruences (G^2, θ, Δ), the corresponding block is a, a sublattice of $Cw\mathcal{A}$ (a two-element chain, or a four-element boolean lattice, see Fig. 1 b)). However, G fails on the CIP (namely on the weak congruences $\rho_1 = \{a, b\}^2$ and $\rho_2 = \{c, d\}^2$).

	a	b	c	d
a	b	b	c	d
b	b	a	c	b
c	c	c	d	c
d	a	b	c	c

$G = \{a, b, c, d\}$
$B = \{a, b\}$
$C = \{c, d\}$ $\quad G/\theta = \{\{a, b\}, \{c\}, \{d\}\}$

a)



b)

Figure 1

In order to give necessary and sufficient conditions for an algebra $\mathcal{A}$ to have the CIP (in terms of particular sublattices of $Cw\mathcal{A}$), we have to introduce the following enlargement of the blocks C_θ. For $\theta \in Con\mathcal{A}$, let

$$C_{\theta\uparrow} := \{\rho \in Cw\mathcal{A} \mid \rho_A \geq \theta\}. \tag{4}$$

Theorem 1 *An algebra $\mathcal{A}$ has the CIP if and only if for every congruence θ on $\mathcal{A}$, the set $C_{\theta\uparrow}$ is a lattice under inclusion, a sublattice of $Cw\mathcal{A}$.*

Proof. Suppose that $\mathcal{A}$ has the CIP, and let $\rho_1, \rho_2 \in C_{\theta\uparrow}$, $\theta \in Con\mathcal{A}$. Then $\rho_1 \vee \Delta \geq \theta$ and $\rho_2 \vee \Delta \geq \theta$, and hence $(\rho_1 \vee \Delta) \wedge (\rho_2 \vee \Delta) \geq \theta$. By the CIP, $(\rho_1 \wedge \rho_2) \vee \Delta \geq \theta$, which proves that $C_{\theta\uparrow}$ is closed under meets. Obviously, it is closed under joins, hence this set is a sublattice of $Cw\mathcal{A}$.

Conversely, suppose that $C_{\theta\uparrow}$ is a sublattice of $Cw\mathcal{A}$. Now, let $(\Delta \vee \rho_1) \wedge (\Delta \vee \rho_2) = \theta$. Then, $\Delta \vee \rho_1 \geq \theta$ and $\Delta \vee \rho_2 \geq \theta$. Since $C_{\theta\uparrow}$ is a lattice, we have $\Delta \vee (\rho_1 \wedge \rho_2) \geq \theta$. The opposite inequality always holds, hence $\Delta \vee (\rho_1 \wedge \rho_2) = (\Delta \vee \rho_1) \wedge (\Delta \vee \rho_2)$, and the CIP holds. □

More detailed description of the sets $C_{\theta\uparrow}$ is the following.

Proposition 2 *If $\mathcal{A}$ has the CIP, then for every $\theta \in Con\mathcal{A}$, the set $C_{\theta\uparrow}$ is a filter in the lattice $Cw\mathcal{A}$; this filter is principal if and only if $\mathcal{A}$ has the $*$CIP.*

Proof. If $\rho \in C_{\theta\uparrow}$ and for $\sigma \in Con\mathcal{A}$, $\rho \leq \sigma$, then obviously $\sigma \in C_{\theta\uparrow}$, by the definition of the latter. Further, if $\rho_1, \rho_2 \in C_{\theta\uparrow}$, then, since $\mathcal{A}$, has the CIP, $\rho_1 \wedge \rho_2 \in C_{\theta\uparrow}$ by the previous theorem. Hence, $C_{\theta\uparrow}$ is a filter in $Cw\mathcal{A}$.

It was proved in [4] that $\mathcal{A}$ has the $*$CIP iff for every $\theta \in Con\mathcal{A}$, the set $C_{\theta\uparrow}$ has the least element. This proves the second part. □

Getting back to the congruence induced on $Cw\mathcal{A}$ by n_Δ if $\mathcal{A}$ has the CIP, we show that its blocks C_θ, $\theta \in Con\mathcal{A}$ are closely (namely, by a homomorphism) related to the subalgebra lattice of $\mathcal{A}$.

Proposition 3 *If $\mathcal{A}$ has the CIP, then for each $\theta \in Con\mathcal{A}$, the map $\phi : \rho \mapsto \rho \wedge \Delta$ (in $Cw\mathcal{A}$) is a homomorphism from C_θ to $Sub\,\mathcal{A}$.*

Proof. The map ϕ is a restriction to C_θ of the endomorphism $m_\Delta : \rho \mapsto \rho \wedge \Delta$ which maps $Cw\mathcal{A}$ into its ideal $\Delta\downarrow$. Since C_θ is a sublattice of $Cw\mathcal{A}$, ϕ is a homomorphism. □

Recall that an algebra $\mathcal{A}$ has the CEP if and only if Δ is a cancellable element in $Cw\mathcal{A}$ i.e., iff from $\rho_1 \wedge \Delta = \rho_2 \wedge \Delta$ and $\rho_1 \vee \Delta = \rho_2 \vee \Delta$, it follows that $\rho_1 = \rho_2$ ([4]).

Proposition 4 *If $\mathcal{A}$ has the CIP and the CEP, then for every $\theta \in Con\mathcal{A}$, the lattice C_θ is embeddable into $Sub\mathcal{A}$.*

Proof. If ; $\mathcal{A}$ has the CIP, then by the foregoing proposition $\phi : \rho \mapsto \rho \wedge \Delta$ is a homomorphism from C_θ to $\Delta\downarrow$, i.e., to $Sub\mathcal{A}$. Further, ϕ is an injection by the CEP. Indeed, if $\rho_1, \rho_2 \in C_\theta$, then $\rho_1 \vee \Delta = \rho_2 \vee \Delta$. Now, if $\rho_1 \wedge \Delta = \rho_2 \wedge \Delta$, then by the CEP, $\rho_1 = \rho_2$. Hence, ϕ is an embedding. □

We can reformulate the previous proposition in the following way.

Corollary 1 *If $\mathcal{A}$ has the CIP and the CEP, then every congruence θ on $\mathcal{A}$ determines a lattice C_θ of subalgebras of $\mathcal{A}$: if $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, then*

$B \in C_\theta$ if and only if there is $\rho \in Con\mathcal{B}$ such that $\rho \vee \Delta = \theta$.

This lattice has a minimal element if and only if $\mathcal{A}$ has the $*$CIP.

Proof. Straightforward, by the previous proposition and by the definition of the $*$CIP. □

The foregoing result establishes a particular algebraic duality with the well known fact that *to every subalgebra $\mathcal{B}$ of an algebra $\mathcal{A}$, one can associate its* *congruence lattice ConB*. Similarly, if $\mathcal{A}$ has the CIP, then *to every congruence θ on $\mathcal{A}$, one can associate its subalgebra lattice* $\phi(C_\theta)$ (i.e., the lattice of diagonal relations of weak congruences belonging to C_θ). If, in addition, $\mathcal{A}$ has the CEP, then ϕ is an embedding, and the above subalgebra lattice associated to θ is C_θ.

A characterization of particular algebras satisfying the CIP and $*$CIP can be given in terms of homomorphic images of $\mathcal{A}$.

A variety $\mathcal{V}$ is said to be a 0_1-**variety**, if it has a nullary operation 0 in its similarity type, and if every algebra of $\mathcal{V}$ has a one-element subalgebra ([3]). Recall that an algebra $\mathcal{A}$ is **Hamiltonian** if every subalgebra of $\mathcal{A}$ is a block of some congruence of $\mathcal{A}$. A variety consisting of Hamiltonian algebras is **Hamiltonian**. An algebra with a nullary operation 0 is *weakly coherent* [1,2], if for every subalgebra $\mathcal{B}$ of $\mathcal{A}$ and each $\theta \in Con\mathcal{A}$, if $[0]_\theta \subseteq B$, then $[x]_\theta \subseteq B$, for each $x \in B$ (where for $x \in A$, $[x]_\theta$ is a block of the congruence θ to which x belongs).

An algebra $\mathcal{A}$ is said to have the **Strong Congruence Extension Property** (SCEP) ([1]) if for every congruence ρ on a subalgebra $\mathcal{B}$ of $\mathcal{A}$, there is a congruence θ on $\mathcal{A}$ such that $[b]_\rho = [b]_\theta$ for every $b \in B$.

Lemma 1 ([1]) *A variety $\mathcal{V}$ has the SCEP if and only if it is Hamiltonian.*

□

Recall that for $B \in Sub\mathcal{A}$, $\theta \in Con\mathcal{A}$,

$$B[\theta] = \{x \in A \mid x\theta b \text{ for some } b \in B\}.$$

Lemma 2 *An algebra $\mathcal{A}$ has the SCEP if and only if $B[\theta] = B$ for every $\theta \in Con\mathcal{A}$ such that there is $\rho \in Con\mathcal{B}$ satisfying $\rho \vee \Delta = \theta$.*

Proof. Straightforward, by the definition of the SCEP. □

The following theorem is an improvement of some results from [5], hence a part of the proof is taken from there. The present theorem also generalizes the following well known fact: For any algebra $\mathcal{A}$ and $\theta \in Con\mathcal{A}$, the lattice $Con\mathcal{A}/\theta$ is isomorphic with the principal filter $\theta\!\uparrow$ in $Con\mathcal{A}$.

Theorem 2 *Let $\mathcal{V}$ be a Hamiltonian 0_1-variety and $\mathcal{A}$ a weakly coherent algebra from $\mathcal{V}$. Then, $\mathcal{A}$ has the $*$CIP if and only if for every $\theta \in Con\mathcal{A}$, $Cw\mathcal{A}/\theta$ is isomorphic with $C_{\theta\uparrow}$ under $\rho \mapsto \rho/\theta$.*

Proof. Suppose that $\mathcal{A}$ has the $*$CIP. Then, by Theorem 1, $\theta\!\uparrow$ is a sublattice of $Cw\mathcal{A}$. Recall the above mentioned fact that the congruence lattice of the factor algebra is isomorphic with a particular filter in $Con\mathcal{A}$. Further, the weak congruence lattice of an algebra is the union of all congruences on all its subalgebras. Hence, the congruence lattice of any subalgebra of $\mathcal{A}/\theta$ is, up to the above isomorphism, the interval $[B^2 \wedge \theta, B^2]$ in $Cw\mathcal{A}$, where $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ satisfying $B[\theta] = B$. Hence, we have to prove that the lattice $\theta\!\uparrow$ consists precisely of these intervals i.e, that

$$\theta\!\uparrow = \bigcup([B^2 \wedge \theta] \mid B[\theta] = B, B \in Sub\mathcal{A}). \tag{5}$$

If $\rho \in [B^2 \wedge \theta, B^2]$ for some $B \in Sub\mathcal{A}$ such that $B[\theta] = B$, then obviously $\rho \in \theta\!\uparrow$, since $\rho \geq B^2 \wedge \theta \in \theta\!\uparrow$.

Conversely, let $\rho \in \theta\!\uparrow$, $\rho \in Sub\mathcal{B}$ for some subalgebra $\mathcal{B}$ of $\mathcal{A}$. Then, $B^2 \vee \Delta \geq \rho \vee \Delta \geq \theta$. Since $\mathcal{V}$ is Hamiltonian, by Lemma 1, it has the SCEP, and thus $B[\theta] = B$. We prove that $\rho \geq B^2 \wedge \theta$. By the CIP,

$$(B^2 \wedge \rho \wedge \theta) \vee \Delta = (\rho \wedge \theta) \vee \Delta = (\rho \vee \Delta) \wedge \theta = \theta.$$

Again by the CIP,

$$(B^2 \wedge \theta) \vee \Delta = (B^2 \vee \Delta) \wedge \theta = \theta.$$

Hence, $\rho \wedge \theta$ and $B^2 \wedge \theta$ both belong to C_θ, and as congruences on the same subalgebra $\mathcal{B}$ of $\mathcal{A}$, they are equal by the CEP (which holds in any Hamiltonian variety):

$$\rho \wedge \theta = B^2 \wedge \theta.$$

Obviously, $\rho \geq B^2 \wedge \theta$, which proves that $\rho \in [B^2 \wedge \theta, B^2]$. Thus, (5) holds.

Suppose now that for every $\theta \in Con\mathcal{A}$, $Cw\mathcal{A}/\theta$ is isomorphic with $C_{\theta\uparrow}$. Then $C_{\theta\uparrow}$ is a lattice and $\mathcal{A}$ has the CIP by Theorem 1. Since obviously this lattice has the smallest element, $\mathcal{A}$ has also the $*$CIP, by Proposition 2. □

Remark. If C_wA/θ is isomorphic with $C_{\theta\uparrow}$, then obviously $\theta\uparrow$ is a isomorphic with $ConA/\theta$, and C_θ with $SubA/\theta$.

An algebra $\mathcal{A}$ satisfies the **Unique Congruence Extension Property** (the UCEP) ([1]) if for every congruence ρ on a subalgebra of $\mathcal{A}$ there is a unique congruence on $\mathcal{A}$ collapsing ρ.

Lemma 3 ([3]) *An algebra $\mathcal{A}$ has the UCEP if and only if for every congruence ρ on a subalgebra B of A,*

$$B^2 \wedge \theta = \rho \text{ if and only if } \theta = \rho \vee \Delta.$$

□

Lemma 4 ([3]) *The following are equivalent for an algebra $\mathcal{A}$ with a nonempty least subalgebra:*

(1) $\mathcal{A}$ satisfies the UCEP;

(2) $\mathcal{A}$ has CEP, CIP, and for every $B \in SubA$, $B^2 \vee \Delta = A^2$ in C_wA;

(3) $C_wA \cong SubA \times ConA$, under $\rho \mapsto (\rho \wedge \Delta, \rho \vee \Delta)$. □

Algebraic aspects of the duality between subalgebra and congruence lattices appear also in the class of algebras having the UCEP. To show this, we characterize these algebras in terms of classes C_θ, $\theta \in ConA$.

Theorem 3 *An algebra $\mathcal{A}$ with a nonempty least subalgebra has the UCEP if and only if for every $\theta \in ConA$, C_θ is (as a poset ordered by inclusion) isomorphic with $SubA$ under $\rho \mapsto \rho \wedge \Delta$.*

Proof. Let $\mathcal{A}$ be an algebra having the UCEP. Then, by Lemma 4 (2), $\mathcal{A}$ has the CEP and the CIP. Hence, by Proposition 4, $f : \rho \mapsto \rho \wedge \Delta$ is an embedding from C_θ into $SubA$, for every $\theta \in ConA$. This function (f) is a surjection. To show this, we prove that for every $B \in SubA$, $f(B^2 \wedge \theta) = B$. This equality is obvious (not that the least subalgebra is not empty), however we have to prove that $B^2 \wedge \theta \in C_\theta$. Indeed, by the CEP and by (2) from Lemma 4, $(B^2 \wedge \theta) \vee \Delta = (\Delta \vee B^2) \wedge \theta = A^2 \wedge \theta = \theta$. Hence, f is an isomorphism.

Conversely, let C_θ be isomorphic with $SubA$ under $\rho \mapsto \rho \wedge \Delta$, for every $\theta \in ConA$. To prove the UCEP, suppose that there are two congruences θ_1 and θ_2, such that for $B \in ConA$, $B^2 \wedge \theta_1 = B^2 \wedge \theta_2 = \rho$. Obviously, $\rho \in ConB$. Without loss of generality, assume that $\theta_2 \leq \theta_1$, and let $\rho_1 \in ConB$, such that $\rho_1 \vee \Delta = \theta_1$. ρ_1 exists, by the assumption that $C_{\theta_1} \cong SubA$. Now, $\rho_1 \leq B^2$ and $\rho_1 \leq \theta_1$, hence $\rho_1 \leq B^2 \wedge \theta_1$. Thereby, $\theta_1 = \rho_1 \vee \Delta \leq \rho \vee \Delta \leq \theta_2$. Thus, $\theta_1 \leq \theta_2$, which proves $\theta_1 = \theta_2$, i.e, the UCEP holds. □

Lemma 4 and Theorem 3 show that the duality introduced by the CIP in the class of UCEP algebras has the following form.

Corollary 2 *An algebra $\mathcal{A}$ satisfies the UCEP iff*

the congruence lattice of each subalgebra of $\mathcal{A}$ is isomorphic to $ConA$ iff

the lattice of subalgebras associated to each congruence θ of $\mathcal{A}$ (by the class C_θ) is $SubA$. □

Following the result from [6], as an example we have that every algebra with a boolean lattice of weak congruence is a UCEP algebra, hence it satisfies the above listed dual properties.

3 Appendix

For the most of the statements presented above there are lattice theoretic analogons. Since these results have their own meaning and importance in lattice theory, we give them in the sequel. Proofs can be obtained mostly by reformulation of the proofs for the corresponding algebraic propositions.

I *If a is a distributive element of the lattice L, then the mapping $\varphi_a : x \to x \vee a$ is an endomorphism of L into the ideal $a\downarrow$. Consequently, the relation $\sim_a$ on L, defined with $x \sim_a y \longleftrightarrow x \vee a = y \vee a$ is a congruence relation on L, and L/φ_a is isomorphic with the principal filter $a\uparrow$ under φ_a.*

II *If a is distributive in L, then blocks of the congruence $\sim_a$ have maximal elements. The set of maximal elements is the filter $a\uparrow$.*

Distributivity of $a \in L$ does not guarantee the existence of minimal elements in blocks of the congruence $\sim_a$.

III *If a is an infinitely distributive element of the lattice L, then blocks of the congruence $\sim_a$ have minimal elements.*

Remark. Since a congruence splits a lattice into sublattices, it is obvious that minimal (if any) as well as maximal elements in blocks of $\sim_a$ are unique.

IV *Let a be a distributive element of L, and L_1 a sublattice of L which is a block of $\sim_a$. Then, the map $\phi : x \to x \wedge a$ is a homomorphism of L_1 into the sublattice-ideal $a\downarrow$.*

If a is a neutral element of L, then the above map ϕ is an embedding.

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ТРИГОНОМЕТРИИ С ФУНКЦИЯМИ SIN И COS[1]

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В настоящей работе рассматривается аксиоматика тригонометрий пар групп, позволяющая определять аналоги основных тригонометрических функций.

Без определения будут использоваться понятия и обозначения из [1, 2].

Тригонометрия trm пары групп $(\langle G_1,+\rangle, \langle G_2,\cdot\rangle)$ на плоскости, где $R = \langle G_1,+,\cdot\rangle$ — кольцо с единицей 1, называется *тригонометрией, обладающей функциями* Sin *и* Cos, или SC-*тригонометрией*, если существует элемент $\alpha_\perp \in G_2$, имеющий порядок 2 и такой, что

1) если $f_{ra}(a_1,\alpha,1) = \alpha_\perp$ и $f_{ra}(a_2,\alpha,1) = \alpha_\perp$, то $a_1 = a_2$;

2) для любого $\alpha \in G_2 \setminus \{e\}$ тогда и только тогда существует элемент a, для которого $f_{ra}(a,\alpha,1) = \alpha_\perp$, когда $\alpha \neq \alpha_\perp$.

Определим функции $\mathrm{Cos} : G_2 \to G_1$ и $\mathrm{Sin} : G_2 \to G_1$ в SC-тригонометрии trm по следующим правилам:

а) $\mathrm{Cos}\, e = 1$, $\mathrm{Sin}\, e = 0$, $\mathrm{Cos}\, \alpha_\perp = 0$, $\mathrm{Sin}\, \alpha_\perp = 1$,

б) если $\alpha \notin \{e, \alpha_\perp\}$ и $f_{ia}(a,\alpha,1) = \alpha_\perp$, то $\mathrm{Cos}\, \alpha = a$, $\mathrm{Sin}\, \alpha = f_{ms}(a,\alpha,1)$.

Отметим, что функции Cos и Sin являются инъекциями.

Будем говорить, что SC-тригонометрия trm обладает *свойством подобия*, если R — коммутативное кольцо и для любого $c \in G_1 \setminus \{0\}$ из $\begin{pmatrix} a & 1 & b \\ \alpha & \beta & \alpha_\perp \end{pmatrix} \in \mathbf{S}_3(trm)$ следует $\begin{pmatrix} ac & c & bc \\ \alpha & \beta & \alpha_\perp \end{pmatrix} \in \mathbf{S}_3(trm)$.

Непосредственно проверяются следующие свойства SC-тригонометрий, обладающих свойством подобия.

[1] Работа выполнена при финансовой поддержке Российского фонда фундаментальных исследований (96-01-01675)

1. Если $\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \in \mathbf{S}_3(trm)$, $\alpha, \gamma \notin \{e, \alpha_\perp\}$, то существуют единственные элементы $a_1, a_2 \in G_1$, $\beta_1, \beta_2 \in G_2$ такие, что $a = a_1 + a_2$, $\beta = \beta_1 \cdot \beta_2$, $\begin{pmatrix} a_1 & b & f_{ms}(a_1, \alpha, b) \\ \alpha & \beta_1 & \alpha_\perp \end{pmatrix} \in \mathbf{S}_3(trm)$ и $\begin{pmatrix} c & a_2 & f_{ms}(c, \gamma, a_2) \\ \gamma & \alpha_\perp & \beta_2 \end{pmatrix} \in \mathbf{S}_3(trm)$.

2. Если $\begin{pmatrix} a & b & c \\ \alpha & \beta & \gamma \end{pmatrix} \in \mathbf{S}_3(trm)$, $d \in G_1 \setminus \{0\}$, то $\begin{pmatrix} ad & bd & cd \\ \alpha & \beta & \gamma \end{pmatrix} \in \mathbf{S}_3(trm)$.

3. Если $\begin{pmatrix} a & c & b \\ \alpha & \beta & \alpha_\perp \end{pmatrix} \in \mathbf{S}_3(trm)$, то $a = c \cdot \operatorname{Cos}\alpha$, $b = c \cdot \operatorname{Sin}\alpha$, $b = f_{ls}(\alpha_\perp, a, \alpha) \cdot \operatorname{Sin}\alpha$, $a = f_{rs}(\beta, b, \alpha_\perp) \cdot \operatorname{Sin}\ (\beta^{-1})$. В частности, $\operatorname{Cos}\alpha = \operatorname{Sin}(\beta^{-1})$, $\operatorname{Sin}\alpha = \operatorname{Cos}(\beta^{-1})$.

Тригонометрическая алгебра, соответствующая SC-тригонометрии, называется *SC-тригонометрической алгеброй.*

Очевидно, что условия, определяющие SC-тригонометрии, а также свойство подобия, аксиоматизируются формулами первого порядка на языке тригонометрических алгебр. Таким образом, класс всех SC-тригонометрических алгебр (со свойством подобия) аксиоматизируем.

Тригонометрия $trm(G_1, G_2, \langle P, L, \in \rangle)$ называется *правильной*, если система $\langle P, L, \in \rangle$ удовлетворяет следующей формуле:

$$\forall p_1, p_2, p_3, p_4 \in P\ (\bigwedge_{i \neq j} p_i \neq p_j \to \exists p_5 \in P\ (\bigwedge_{i=1}^{4} p_5 \neq p_i \wedge$$
$$\wedge (\bigvee \{p_5 \in l(p_i, p_j) \cap l(p_m, p_n) \mid \{i, j, m, n\} = \{1, 2, 3, 4\}\})\ ))\ .$$

Из [2, §3] следует

ТЕОРЕМА 1. *Теория класса всех тригонометрических алгебр, соответствующих правильным SC-тригонометриям (со свойством подобия), подобна конечно аксиоматизируемой теории.*

Тригонометрия $trm(\langle G_1, + \rangle, \langle G_2, \cdot \rangle, \mathcal{P})$, где $R = \langle G_1, +, \cdot \rangle$ — кольцо, называется *тригонометрией пары (кольцо, группа).*

ТЕОРЕМА 2. *Любая тригонометрия пары (кольцо с единицей, группа) на плоскости вложима в некоторую SC-тригонометрию.*

Доказательство. Пусть trm — тригонометрия на плоскости пары (R, G_2), где R — кольцо с единицей. Обозначим через λ мощность $\max\{|R|, |G_2|, \omega\}$. Рассмотрим свободное λ-порожденное кольцо R_λ с единицей и со свободными порождающими r_i, $i \in \lambda$, а также свободную λ-порожденную группу F_λ со свободными порождающими g_i, $i \in \lambda$. Покажем, что тригонометрия trm вложима в некоторую SC-тригонометрию trm' пары $(R', G'_2) = (R \times R_\lambda, G_2 * F_\lambda * \mathbf{Z}_2)$, где $\alpha_\perp$ — неединичный элемент группы $\mathbf{Z}_2$.

В дальнейшем для простоты будем предполагать, что $\lambda = \omega$. Пусть $\nu : \omega \to X$, где $X = ((R' \setminus \{0\}) \times (G'_2 \setminus \{e\}) \times (R' \setminus \{0\})) \cup (G'_2 \setminus \{e, \alpha_\perp\})$, — нумерация множества X. Построим семейство тригонометрий trm_n, $n \in \omega \cup \{-1\}$. Обозначим через trm_{-1} связную тригонометрию, задаваемую соотношением:

$$\mathbf{S}(trm_{-1}) = GN(\mathbf{S}(trm) \cup \mathbf{\Delta}_c(R', G'_2)) \setminus \mathbf{S}^e(R', G'_2).$$

Если тригонометрия trm_n уже построена, $\nu(n+1) = (a, \alpha, b)$, и $\begin{pmatrix} a & b & \cdot \\ \alpha & \cdot & \cdot \end{pmatrix} \in \mathbf{S}_3(trm_n)$, то положим $trm_{n+1} \rightleftharpoons trm_n$. Если $\nu(n+1) = (a, \alpha, b)$ и $f_{la}(a, \alpha, b)$ не определено, то рассмотрим наименьшие порождающие r_i, g_j, g_{j+1}, которые не участвуют в разложениях параметров нетривиальных треугольников из trm_n на порождающие. Определим связную тригонометрию trm_{n+1} следующим равенством:

$$\mathbf{S}(trm_{n+1}) = GN\left(\mathbf{S}(trm_n) \cup \left\{ \begin{pmatrix} a & b & r_i \\ \alpha & g_j & g_{j+1} \end{pmatrix} \right\}\right) \setminus \mathbf{S}^e(R', G'_2).$$

Если $\nu(n+1) = \alpha$ и $f_{ra}(a, \alpha, 1) = \alpha_\perp$ для некоторого a, то $trm_{n+1} \rightleftharpoons trm_n$. Если же $f_{la}(a, \alpha, b)$ не определено, то рассмотрим наименьшие порождающие r_i, r_{i+1}, g_j, которые не участвуют в разложениях параметров нетривиальных треугольников из trm_n на порождающие. Определим связную тригонометрию trm_{n+1} следующим соотношением:

$$\mathbf{S}(trm_{n+1}) = GN\left(\mathbf{S}(trm_n) \cup \left\{ \begin{pmatrix} r_i & 1 & r_{i+1} \\ \alpha & g_j & \alpha_\perp \end{pmatrix} \right\}\right) \setminus \mathbf{S}^e(R', G'_2).$$

Искомая связная тригонометрия trm' задается следующим равенством: $\mathbf{S}(trm') = \cup_{n\in\omega}\mathbf{S}(trm_n)$. Теорема доказана.

Отметим, что приведенное доказательство позволяет построить SC-тригонометрию относительно любого семейства элементов вида $\alpha_\perp$.

В заключение сформулируем

Вопрос. Существует ли тригонометрия пары (коммутативное кольцо с единицей, группа), обладающая свойством подобия и не вложимая ни в какую SC-тригонометрию со свойством подобия ?

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ON REPRESENTATION OF LATTICES BY WEAK CONGRUENCES AND WEAK TOLERANCES

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1. Introduction

A **weak congruence** relation ρ on an algebra $\mathcal{A} = (A, F)$ is a symmetric and transitive relation on A compatible with all the operations from F, including nullary ones (if c is a nullary operation, then $c\rho c$). The set of all the weak congruences is an algebraic lattice under inclusion (usually denoted by $Cw\mathcal{A}$). A^2, B^2, the diagonal relations Δ and $\Delta_B = \{(x,x) \mid x \in B\}$, for B subalgebra of $\mathcal{A}$, are weak congruences for every algebra $\mathcal{A}$. Furthermore, in a weak congruence lattice $Cw\mathcal{A}$, the filter $\uparrow\Delta$ is the congruence lattice ($Con\mathcal{A}$) and the ideal $\downarrow\Delta$ is isomorphic with the subuniverse lattice ($Sub\mathcal{A}$) of the algebra $\mathcal{A}$. If $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, then the interval $[\Delta_B, B^2]$ is the congruence lattice $Con\mathcal{B}$. Therefore, the weak congruence lattice is the union of all the congruences on all the subalgebras, plus $\emptyset$ if there are no nullary operations in F. In papers [8-12] by weak congruence lattice is considered the union of all the congruences on all the subalgebras of an algebra, where $\emptyset$ is added if and only if that algebra does not have the least subalgebra (in order to complete the family of all the congruences on all the subalgebras to be a lattice).

Let ρ be a relation on an algebra $\mathcal{A}$. ρ is **weakly reflexive** if for every $x, y \in A$

$$\rho(x,y) \Rightarrow \rho(x,x) \wedge \rho(y,y).$$

ρ is a **tolerance relation** if it is a reflexive, symmetric and compatible relation on $\mathcal{A}$. A **weak tolerance relation** is weakly reflexive, symmetric and compatible.

Denote by $TolA$, TwA, $RefA$, RwA classes of all tolerance relations, all weak tolerance relations, all reflexive compatible relations and all weakly reflexive compatible relations on an algebra $\mathcal{A}$, respectively.

All of the mentioned sets of relations are algebraic lattices under inclusion. All of them include Δ (the diagonal relation) and A^2. TwA and RwA also include B^2 and Δ_B, for all the subalgebras $\mathcal{B}$ of $\mathcal{A}$.

Recall that an element $a \in L$ is codistributive if it satisfies $a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$, for all $x, y \in L$. An element satisfying the dual identity is called distributive. If L is algebraic lattice, then a codistributive element a is infinite codistributive as well, and classes of the congruence corresponding to the homomorphism $n_\Delta : x \mapsto x \wedge \Delta$ always have top elements. We denote by b^* the top element of the class to which an element b belongs.

An element $a \in L$ is cancellable if from $x \wedge a = y \wedge a$ and $x \vee a = y \vee a$ it follows that $x = y$, for all $x, y \in L$. An element $a \in L$ is neutral if it is distributive, codistributive and cancellable. The center of a lattice is the set of all the neutral elements having complements.

The diagonal relation Δ is always a codistributive element in lattices CwA, TwA and RwA. In all these lattices the top elements of the classes of the congruences determined by n_Δ are all squares of all the subalgebras of $\mathcal{A}$.

Recall that an algebra $\mathcal{A}$ satisfies the **Congruence Extension Property (CEP)** if every congruence on a subalgebra has a congruence which is an extension of that congruence. Algebra $\mathcal{A}$ satisfies the CEP if and only if the diagonal relation Δ is a cancellable element in the lattice CwA. An algebra $\mathcal{A}$ satisfies the **Unique Congruence Extension Property (UCEP)** if every congruence ρ on a subalgebra of $\mathcal{A}$ has a unique congruence on $\mathcal{A}$ extending it. An algebra satisfies the UCEP if and only if $CwA \cong SubA \times ConA$, provided that $\mathcal{A}$ contains a minimal subalgebra [8], if and only if Δ is in the center of CwA. An algebra $\mathcal{A}$ satisfies the **Congruence Intersection Property (CIP)** if Δ is a distributive element in CwA. Similarly, an algebra satisfies the **Tolerance Extension Property (TEP)** if the diagonal relation is a cancellable element in the weak tolerance lattice, and the **Tolerance Intersection Property (TIP)** if Δ is a distributive element in TwA.

2. Preliminaries

Grätzer-Schmidt theorem on representation of lattices by congruences is well known. In [3] Chajda and Czédli proved that any algebraic lattice is isomorphic to $TolA$ for an algebra $\mathcal{A}$ such that every compatible binary relation on $\mathcal{A}$ is a tolerance on $\mathcal{A}$. This shows that for every algebraic lattices L there is an algebra $\mathcal{A}$ such that $L \cong TolA$ and $L \cong RefA$.

In [12] Vojvodić and Šešelja note that for every algebraic lattice there is an algebra $\mathcal{A}$ such that CwA is isomorphic with that lattice. Namely, for a lattice L we take algebra $\mathcal{B} = (A, F)$ such that $ConB$ is isomorphic with L, which exists by Grätzer-Schmidt theorem, and construct the required algebra $\mathcal{A}$ by adding to F all the elements from A as nullary operations. By the similar arguments from the mentioned theorem by Chajda-Czédli one can deduce that for every algebraic lattice there is an algebra such that TwA and RwA are isomorphic with that lattice.

There is still an open problem of representation of lattices by weak congruences (weak tolerances and weak reflexive compatible relations), with a fixed element of a lattice representing the diagonal relation.

In order to solve this problem for weak congruences, Vojvodić and Šešelja in [12] introduce the notion of a Δ-suitable element. Let L be an algebraic lattice. Element a is **Δ -suitable** if it satisfies the following conditions:

(1) a is codistributive element in L;

(2) For all $x, y \in L$, if $x \wedge y \neq 0$ then $(x \vee y)^* = x^* \vee y^*$;

(3) For all $x, y \in L$, if $x \neq 0$ and $x^* < y$, then $(y \wedge a)^* \neq y \wedge a$;

(4) If $x \neq 0$ and $x = \bigwedge(y \mid y \in L \setminus \{0\})$, then $x \neq x^*$, for every $x \in L$;

(5) If $x \in \downarrow a$ and $x \prec a$, then

$$\bigvee(y \in \uparrow a \mid y \vee x^* < 1) \neq 1.$$

They proved that if f is a lattice isomorphism from CwA to a lattice L, then $f(\Delta)$ is a Δ-suitable element.

M. Ploščica generalized the condition (5) in the following way: [1]

(5') If $x, y \in \downarrow a$, and $x \prec y$, then it exists $z \in [y, y^*]$, such that

[1] Private correspondence.

(*i*) for all $t \in [x, x^*]$, the set

$$\{c \in Ext^y(t) \mid c \leq z\}$$

is either empty or possesses the top element;

(*ii*) for all $t \in [x, x^*]$, the set

$$\{c \in Ext^y(t) \mid c \not\leq z\}$$

is an antichain (possibly empty),
where

$$Ext^y(t) = \{w \in [y, y^*] \mid w \cap x^* = t\}.$$

Since a Δ-suitable element is codistributive, it is infinite codistributive as well and classes of the congruence determined by the homomorphism n_a always have top elements.

In [7] M. Ploščica gave a solution of the concrete representation problem for weak congruence lattices, i.e., he answered the question: when a subfamily of the lattice of weak equivalences on A is the set of all the weak congruences on an algebra $\mathcal{A} = (A, F)$. Namely, a family $\mathcal{F} \subseteq Ew(A)$ is the set of all the weak congruences of some algebra if and only is $\mathcal{F}$ is closed under all the graphical compositions and up-directed unions (for details, see [7]).

3. Results

In this section the problem of representation of lattices by weak congruences, weak tolerances and weak reflexive compatible relations with a fixed element of a lattice representing the diagonal relation is solved for a class of lattices, the diagonal relation being the image of an element from the center of the lattice we are trying to represent. Since the diagonal relation belongs to the center of the lattice, the obtaining algebra whose lattice of weak congruences is isomorphic with that lattice satisfies the CEP and the CIP. Moreover it satisfies the UCEP as well. Similarly, the obtaining algebras for lattices of weak tolerances and weakly reflexive compatible relations satisfy the corresponding extension and intersection properties.

Theorem 1. *Let $\mathcal{L}$ be an algebraic lattice and $a \in L$ an element from the center of the lattice, such that $\uparrow a$ has exactly one coatom. Then, there is an algebra $\mathcal{A}$, such that the weak congruence lattice $Cw\mathcal{A}$ is isomorphic with $\mathcal{L}$ under a mapping f, and $f(\Delta) = a$.*

Proof. Let 1 be the top element of the lattice $\mathcal{L} = (L, \wedge, \vee)$. Let $\mathcal{B} = (B, G)$ be an algebra such that $\downarrow a \cong Sub\mathcal{B}$. Further, let $\mathcal{C} = (C, H)$ be an algebra, for which $\uparrow a \setminus \{1\} \cong Con\mathcal{C}$ and $B \cap C = \emptyset$. Such algebras exist by Birkhoff-Frink and Grätzer-Schmidt theorems, respectively. Required algebra $\mathcal{A}$ such that $Cw\mathcal{A}$ is isomorphic with L is :

$\mathcal{A} = (A, F)$, where $A = B \cup C$, and $F = G^* \cup H^* \cup \{c_x \mid x \in C\} \cup \{s\}$, the operations being as follows. For every $g \in G$, the corresponding operation in G^* is:

$$g^*(x_1, ..., x_n) = \begin{cases} g(x_1, ..., x_n), & x_1, ..., x_n \in B; \\ x_1, & \text{otherwise.} \end{cases}$$

Similarly, for every $h \in H$, the corresponding operation in H^* is:

$$h^*(x_1, ..., x_n) = \begin{cases} h(x_1, ..., x_n), & x_1, ..., x_n \in C; \\ x_1, & \text{otherwise.} \end{cases}$$

For every $x \in C$, c_x is the corresponding nullary operation of $\mathcal{A}$.
Finally, s is the following operation of arity 4:

$$s(x, y, z, t) = \begin{cases} z, & \text{for } x = y \text{ and } x \in B; \\ t, & \text{for } x \neq y \text{ and } x \in B; \\ x, & \text{for } x \in C. \end{cases}$$

Now, we are going to prove that:

- $Sub\mathcal{B} \cong Sub\mathcal{A}$;
- $Con\mathcal{C} \cong Con\mathcal{D} \setminus \{D^2\}$, for every subalgebra $\mathcal{D}$ of $\mathcal{A}$.

All the elements from C must be in all the subuniverses of $\mathcal{A}$ (nullary operations), and all the subuniverses of $\mathcal{A}$ are exactly of the form $C \cup X$, where X is a subuniverse of $\mathcal{B}$. Namely, if X is a subuniverse of $\mathcal{B}$, then $C \cup X$ is obviously closed under all the operations from G^*, H^* and s as well. Conversely, if A_1 is a subuniverse of $\mathcal{A}$, then $A_1 = C \cup X$, for $X \subseteq B$. Since A_1

is closed under all the operations from F, especially under all the operations from G^*, the set X is also closed under all the operations from G, and it is a subuniverse of B.

All the congruences on $\mathcal{A}$, except A^2, are exactly of the form $\rho \cup \Delta_B$, where ρ is a congruence on C. Indeed, if ρ is a congruence on C, then $\rho \cup \Delta_B$ is compatible with all the operations from F, the proof of which follows.

Let $g^* \in G^*$ be an n-ary operation ($n \geq 1$), and $x_1, ..., x_n, y_1, ..., y_n \in A$, such that $x_i(\rho \cup \Delta_B)y_i$, for $1 \leq i \leq n$. Then, for every i,

$$x_i \rho y_i \text{ and } x_i, y_i \in C \text{ or } x_i = y_i \text{ and } x_i, y_i \in B.$$

If all elements x_i and y_i belong to B, we have that $g^*(x_1, ..., x_n) = g^*(y_1, ..., y_n)$, i.e., $g^*(x_1, ..., x_n)(\rho \cup \Delta_B)g^*(y_1, ..., y_n)$.

If there is an element x_j not belonging to B, then y_j does not belong to B as well, and

$$g^*(x_1, ..., x_n) = x_1(\rho \cup \Delta_B)y_1 = g^*(y_1, ..., y_n).$$

Similarly we prove that $\rho \cup \Delta_B$ is compatible with an operation $h^* \in H^*$.

We will prove the same for the operation s. Let $x_i(\rho \cup \Delta_B)y_i$, for $i = 1,2,3,4$. If $x_1 \in C$, then $y_1 \in C$ as well, and $s(x_1, x_2, x_3, x_4) = x_1 \rho y_1 = s(y_1, y_2, y_3, y_4)$. If $x_1 \in B$, and $x_1 = x_2$, then $y_1 = x_1$, $y_1 = y_2 = x_2$ and $y_1 \in B$, and hence $s(x_1, x_2, x_3, x_4) = x_3(\rho \cup \Delta_B)y_3 = s(y_1, y_2, y_3, y_4)$. If $x_1 \in B$ and $x_1 \neq x_2$, then $y_1 \neq y_2$, and $s(x_1, x_2, x_3, x_4) = x_4(\rho \cup \Delta_B)y_4 = s(y_1, y_2, y_3, y_4)$.

On the other hand, if θ is a congruence on $\mathcal{A}$, then $\theta \cap C^2$ is a congruence on $\mathcal{C}$. If $\theta \neq A^2$ we will prove that θ is exactly of the form $(\theta \cap C^2) \cup \Delta_B$. Suppose that there are elements a, b such that $a \neq b$, $a \in B$ and $a\theta b$. Then, if x, y are arbitrary elements from A,

$$x = s(a, a, x, y)\theta s(a, b, x, y) = y,$$

which means that $\theta = A^2$.

Similarly, if $\mathcal{D}$ is a subalgebra of $\mathcal{A}$, then all the congruences of $\mathcal{D}$, except D^2, are exactly of the form $\rho \cup \Delta_D$, where ρ is a congruence on $\mathcal{C}$.

Since a belongs to the center of L, $L \cong \uparrow a \times \downarrow a$. Further on, $\downarrow a \cong Sub\mathcal{B}$, $Sub\mathcal{B} \cong Sub\mathcal{A}$ and $Sub\mathcal{A} \cong \downarrow \Delta$ in $Cw\mathcal{A}$. Similarly, $\uparrow a \setminus \{1\} \cong Con\mathcal{C}$, $Con\mathcal{C} \cong$

$Con\mathcal{A} \setminus \{A^2\}$, and $Con\mathcal{A} \cong \uparrow \Delta$ in $Cw\mathcal{A}$. For all $b \in \downarrow a$, $[b, b^*] \setminus \{b^*\} \cong Con\mathcal{C}$. Finally, $Con\mathcal{D} \setminus \{D^2\} \cong Con\mathcal{C}$, for every $\mathcal{D} \in Sub\mathcal{A}$.

Therefore, $Cw\mathcal{A} \cong L$. □

Obviously, we have that in a weak tolerance lattice the filter $\uparrow \Delta$ is the tolerance lattice ($Tol\mathcal{A}$). Similarly in a lattice of all weakly reflexive compatible relations the filter $\uparrow \Delta$ is the lattice $Ref\mathcal{A}$. The ideal $\downarrow \Delta$ is isomorphic with the subuniverse lattice ($Sub\mathcal{A}$) of algebra $\mathcal{A}$ in both, $Tw\mathcal{A}$ and $Rw\mathcal{A}$. If $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, then the interval $[\Delta_B, B^2]$ in $Tw\mathcal{A}$ is the tolerance lattice $Tol\mathcal{B}$ (In $Rw\mathcal{A}$ the mentioned interval is $Ref\mathcal{B}$) [4]. Using the cited results by Chajda and Czédli on representation of lattices by tolerances and by lattices of reflexive compatible relations as well as the construction similar to the one in Theorem 1, we get the following results.

Theorem 2. *Let $\mathcal{L}$ be an algebraic lattice and $a \in L$ an element from the center of the lattice, such that $\uparrow a$ has exactly one coatom. Then, there is an algebra $\mathcal{A}$, such that $Tw\mathcal{A} \cong \mathcal{L}$ under a mapping f, and $f(\Delta) = a$.* □

Theorem 3. *Let $\mathcal{L}$ be an algebraic lattice and $a \in L$ an element from the center of the lattice, such that $\uparrow a$ has exactly one coatom. Then, there is an algebra $\mathcal{A}$, such that $Rw\mathcal{A} \cong \mathcal{L}$ under a mapping f, and $f(\Delta) = a$.* □

The following example illustrates the theorem for weak congruences.

Example 1.

Let L and $a \in L$ be as in the lattice in Fig. 1. Element a satisfies the conditions from Theorem 1, and we can represent lattice L by a weak congruence lattice

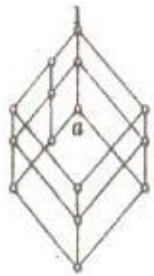

Figure 1

We have to construct an algebra B such that $SubB$ is isomorphic with $\downarrow a$ (pentagon lattice), and an algebra C such that $ConC$ is isomorphic with $\uparrow a \setminus \{1\}$ (two-element chain). Such algebras are: two element group $C = (C, \circ)$, where $C = \{b, c\}$ and a groupoid $B = (B, *)$, where $B = \{d, e, f\}$, and a binary operation $*$ is defined by the following table:

$*$	d	e	f
d	d	e	e
e	d	d	e
f	e	d	f

Table 1

The required algebra is $\mathcal{A} = (A; s, \circ, *, b, c)$, where s is the operation of arity 4, $\circ$ and $*$ are binary operations on A defined as in Theorem 1, using the operations $\circ$ and $*$ on C and B, respectively. Finally, b and c are nullary operations. By Theorem 1, $Cw\mathcal{A} \cong \mathcal{L}$, under the isomorphism f, where $f(\Delta) = a$. $\square$

The similar problem of representation of lattices by weak quasi-orders (weakly reflexive and transitive relations) is not considered here. The reason is that the problem of representation of lattices by quasi-orders is still open [6]. If it happens to be true that for every algebraic lattice there is a lattice of quasi-orders on an algebra isomorphic with this lattice, then the construction similar to the one in Theorem 1 would give representation considered here for weak quasi-orders, as well.

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n-GROUPS, $n \geq 2$, AS VARIETIES OF TYPE $(n, n-1, n-2)$

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1. Preliminaries

Let $p \in \mathrm{N}$, $q \in \mathrm{N} \cup \{0\}$ and let a be a mapping of the set $\{i \mid i \in \mathrm{N} \wedge i \geq p \wedge i \leq q\}$ into the set S; $\emptyset \notin S$. Then:

$$a_p^q \text{ stands for } \begin{cases} a_p, \ldots, a_q; & p < q \\ a_p; & p = q \\ \text{empty sequence } (= \emptyset); & p > q. \end{cases}$$

For example:
$A(a_1^{j-1}, A(a_j^{j+n-1}), a_{j+n}^{2n-1})$, $j \in \{1, \ldots, n\}$, $n \in \mathrm{N} \backslash \{1, 2\}$, for $j = n$ stands for

$$A(a_1, \ldots, a_{n-1}, A(a_n, \ldots, a_{2n-1})).$$

Besides, in some situations *instead of* a_p^q *we write* $(a_i)_{i=p}^q$ (briefly: $(a_i)_p^q$).
For example:

$$(\forall x_i \in Q)_1^q$$

for $q > 1$ stands for

$$\forall x_1 \in Q \ldots \forall x_q \in Q$$

[usually, we write: $(\forall x_1 \in Q) \ldots (\forall x_2 \in Q)$],
for $q = 1$ it stands for

$$\forall x_1 \in Q$$

[usually, we write: $(\forall x_1 \in Q)$],
and for $q = 0$ it stands for an empty sequence $(= \emptyset)$.

In *some cases*, instead of a_p^q only, we write: sequence a_p^q (sequence a_p^q over a set S). For example: ... for every sequence a_p^q over a set S ... . And if $p \leq q$, we usually write: $a_p^q \in S$.

2. Introduction

2.1 On n-groups

2.1.1. Definitions: *Let $n \geq 2$ and let (Q, A) be an n groupoid. Then: (a) we say that (Q, A) is an n-semigroup iff for every $i, j \in \{1, \ldots, n\}$, $i < j$, the following law holds*

$$A(x_1^{i-1}, A(x_i^{i+n-1}), x_{i+n}^{2n-1}) = A(x_1^{j-1}, A(x_j^{j+n-1}), x_{j+n}^{2n-1})$$

[: $\langle i, j \rangle$-associative law]; (b) we say that (Q, A) is an n-quasigroup iff for every $i \in \{1, \ldots, n\}$ and for every $a_1^n \in Q$ there is **exactly one** *$x_i \in Q$ such that the following equality holds*

$$A(a_1^{i-1}, x_i, a_i^{n-1}) = a_n; \text{ and}$$

(c) we say that (Q, A) is a Dörnte n-group [briefly: n-group] iff (Q, A) is an n-semigroup and an n-quasigroup as well.

A notion of an n-group was introduced by W. Dörnte in [1] as a generalization of the notion of a group.

2.1.2. Remark: *n-groups, as varieties , have been investigated, with some exceptions, only for $n \geq 3$ ([14], p. 660; [4 − 8]). In [14] (p. 660) three articles are cited in which n-groups are considered as varieties for $n \geq 2$ ([7 − 9] from [14]; Rusakov S. A. (1979), Dudek W. A. (1980), Ušan J. (1994)).*

In [6] (p. 52), one can find the following view on n-groups: "The theory of n-groups for $n \geq 3$ is essentially different from the theory of groups (i.e. 2-groups), since for $n \geq 3$ **there is no notion which could be an analogue to the neutral element**".

2.2 On the $\{i, j\}$-neutral operation in an n-groupoid

2.2.1. Definitions [10]*: Let $n \geq 2$ and let (Q, A) be an n-groupoid. Further on, Let e_L, e_R and e be mappings of the set Q^{n-2} into the set Q. Let also $\{i, j\} \subseteq \{1, \ldots, n\}$ and $i < j$. Then: a) e_L is a* **left** *$\{i, j\}$-neutral operation of the n-groupoid (Q, A) iff the following formula is satisfied*

$$\text{(a)} \qquad (\forall a_i \in Q)_1^{n-2}(\forall x \in Q) A(a_1^{i-1}, e_L(a_1^{n-2}), a_i^{j-2}, x, a_{j-1}^{n-2}) = x;$$

b) e_R *is a* **right** $\{i,j\}$*-neutral operation of the n-groupoid* (Q,A) *iff the following formula holds*

$$(b) \qquad (\forall a_i \in Q)_1^{n-2}(\forall x \in Q)A(a_1^{i-1}, x, a_i^{j-2}, \mathrm{e}_R(a_1^{n-2}), a_{j-1}^{n-2}) = x; \text{ and}$$

b) e *is an* $\{i,j\}$*-neutral operation of the n-groupoid* (Q,A) *iff the following formula holds*

$$(c) \qquad (\forall a_i \in Q)_1^{n-2}(\forall x \in Q)(A(a_1^{i-1}, \mathrm{e}(a_1^{n-2}), a_i^{j-2}, x, a_{j-1}^{n-2}, = x; \wedge$$

$$\wedge A(a_1^{i-1}, x, a_i^{j-2}, \mathrm{e}(a_1^{n-2}), a_{j-1}^{n-2}) = x). \qquad \text{▫}$$

2.2.2. Remark: *For* $n = 2$, $\mathrm{e}_L(a_1^{n-2}) \models \mathrm{e}_L(\emptyset)$; *1],* $\mathrm{e}_R(a_1^{n-2})$ *and* $\mathrm{e}(a_1^{n-2})$ *are respectively a left neutral element, a right neutral element and a neutral element of the groupoid* (Q,A).

2.2.3. Proposition [10]: *Let* $n \geq 2$, $\{i,j\} \subseteq \{1,\dots,n\}$ *and* $i < j$. *Then: if* e_L *is a* **left** $\{i,j\}$*-neutral operation of the n-groupoid* (Q,A) *and* e_R *is a* **right** $\{i,j\}$*-neutral operation of* (Q,A)*, then* $\mathrm{e}_L = \mathrm{e}_R$ *and* $\mathrm{e} = \mathrm{e}_L = \mathrm{e}_R$ *is an* $\{i,j\}$*-neutral operation of* (Q,A).

The sketch of the part of the proof: If (a) and (b) from 2.2.1 hold, then for every sequence a_1^{n-2} over Q (:1) the following equalities hold

$$A(a_1^{i-1}, \mathrm{e}_L(a_1^{n-2}), a_i^{j-2}, \mathrm{e}_R(a_1^{n-2}), a_{j-1}^{n-2}) = \mathrm{e}_R(a_1^{n-2}) \text{ and}$$

$$A(a_1^{i-1}, \mathrm{e}_L(a_1^{n-2}), a_i^{j-2}, \mathrm{e}_R(a_1^{n-2}), a_{j-1}^{n-2}) = \mathrm{e}_L(a_1^{n-2}).$$

2.2.4. Proposition [10]: *Let* $n \geq 2$, $\{i,j\} \subseteq \{1,\dots,n\}$ *and* $i < j$. *Then in every n-groupoid there is* **at most one** $\{i,j\}$ *neutral operation.*

The sketch of the part of the proof: In the assumption that e_1 and e_2 are $\{i,j\}$-neutral operations of the n-groupoid (Q,A), we take into account only, for instance, the fact that e_1 is a left and e_2 a right $\{i,j\}$-neutral operation, and after that we conclude as in the proof of Proposition 2.2.3.

2.3 On main results of the article

If (Q,A) is an n-group, $n \geq 2$, then $(Q, \overset{2}{A})$, where

$$\overset{2}{A}(x_1^{2n-1}) \overset{def}{=} A(A(x_1^n), x_{n+1}^{2n-1}),$$

is a $(2n-1)$-group [:2.1.1.]. Every m-group, $m \geq 2$, has an $\{1,m\}$-neutral operation [:[10]; 3.4]. Denote by e the $\{1,n\}$-neutral operation of the n-group (Q,A), and by E the $\{1,2n-1\}$-neutral operation of the $(2n-1)$-group $(Q,\overset{2}{A})$. In addition, let

$$(a_1^{n-2}, a)^{-1} \overset{def}{=} \mathsf{E}(a_1^{n-2}, a, a_1^{n-2})\text{[1]}$$

[:3.1-3.4]. In this way, an algebra $(Q, \{A, ^{-1}, \mathrm{e}\})$ of the type $\langle n, n-1, n-2\rangle$ is associated to every n-group (Q,A). Among laws which hold in this algebra, we point out the following ones:

$$(1_L) \quad A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1, A(x_2^{n+1}), x_{n+2}^{2n-1})$$

$$(1_R) \quad A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))$$

$$(2_L) \quad A(\mathrm{e}(a_1^{n-2}), a_1^{n-2}, x) = x$$

$$(2_R) \quad A(x, a_1^{n-2}, \mathrm{e}(a_1^{n-2})) = x$$

$$(3_L) \quad A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathrm{e}(a_1^{n-2})$$

$$(3_R) \quad A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathrm{e}(a_1^{n-2})$$

$$(4_L) \quad A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, x)) = x$$

$$(4_R) \quad A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = x$$

[:3.4]. Further on, among the above laws, we choose six [for $n = 2$ four[2]] systems of three laws, as described in Tables 1-6 [the laws which are represented by marked fields in Tables 1-6]. Now we can formulate basic parts of the main results of the present paper [:Theorems 4.1 and 4.2] in the following way:

Let $n \geq 2$ and let (Q,A) be an n-groupoid. Then: (Q,A) is an n-group iff there are mappings $^{-1}$ and e respectively of the sets Q^{n-1} and Q^{n-2} into the set Q such that in the algebra $(Q, \{A, ^{-1}, \mathrm{e}\})$ of the type $\langle n, n-1, n-2\rangle$ the laws from the marked fields from Table i, $i \in \{1,\dots 6\}$, are satisfied.

[1] For $n = 2$, $a^{-1} = \mathsf{E}(a)$; a^{-1} id the inverse element of the element a with respect to the neutral element of the group (Q,A).

[2] For $n = 2$, $(1_L) = (1_R)$.

	L	R
1		•
2	•	
3	•	
4		

Tabl. 1

	L	R
1	•	
2		•
3		•
4		

Tabl. 2

	L	R
1	•	
2	•	
3		
4		•

Tabl. 3

	L	R
1		•
2	•	
3		
4		•

Tabl. 4

	L	R
1		•
2		•
3		
4	•	

Tabl. 5

	L	R
1	•	
2		•
3		
4	•	

Tabl. 6

3. Auxiliary propositions

3.1. Proposition: *Let $n \geq 2$ and let (Q, A) be an n-groupoid in which the $(1, n)$-associative law holds (:2.1.1). Then, there is* **at most one** *$(n-1)$-ary operation $^{-1}$ in Q such that for every $x, a \in Q$ and for every sequence a_1^{n-2} over Q the following equalities hold*

$$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, x)) = x \text{ and}$$
$$A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = x$$

[:(4$_L$) and (4$_R$) from 2.3].

Proof. Suppose that there are mappings $^{-1_1}$ and $^{-1_2}$ of the set Q^{n-1} into the set Q such that the laws (4_L) and (4_R) from 2.3 hold in the algebras $(Q, \{A, ^{-1_1}\})$ and $(Q, \{A, ^{-1_2}\})$. Then for every $x, y, a \in Q$ and for every sequence a_1^{n-2} over Q (:$n \geq 2$; 1) the following equalities hold

$$A((a_1^{n-2}, a)^{-1_1}, a_1^{n-2}, A(a, a_1^{n-2}, x)) = x \text{ and}$$
$$A(A(y, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1_2}) = y$$

whence, introducing the substitutions $x = (a_1^{n-2}, a)^{-1_2}$ and $y = (a_1^{n-2}, a)^{-1_1}$, by the assumption that in (Q, A) the $(1, n)$-associative law holds, we conclude

that $^{-1_1} = ^{-1_2}$.

3.2. Proposition: *Let $n \geq 2$ and let (Q, A) be an n-groupoid in which the $(1, n)$-associative law holds (:2.1.1). Further on, let (Q, A) has an $\{1, n\}$-neutral operation* **e** *(:2.2.1, 2.2.2, 2.2.4). Then, there is* **at most one** *$(n-1)$-ary operation $^{-1}$ in Q such that for every $x, a \in Q$ and for every sequence a_1^{n-2} over Q the following equalities hold*

$$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2}) \text{ and}$$
$$A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathbf{e}(a_1^{n-2})$$

[:(3$_L$) and (3$_R$) from 2.3].

Proof. Suppose that there are mappings $^{-1_1}$ and $^{-1_2}$ of the set Q^{n-1} into the set Q such that the laws (3_L) and (3_R) from 2.3 hold in the algebras $(Q, \{A, ^{-1_1}, \mathbf{e}\})$ and $(Q, \{A, ^{-1_2}, \mathbf{e}\})$. Then for every $a \in Q$ and for every sequence a_1^{n-2} over Q (:$n \geq 2$; 1) the following equalities hold

$$A((a_1^{n-2}, a)^{-1_1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2}) \text{ and}$$
$$A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1_2}) = \mathbf{e}(a_1^{n-2}),$$

whence, by the monotony of the operation A, by the assumption that the $(1, n)$-associative law holds in (Q,A) and by the assumption that e is an $\{1, n\}$-neutral operation of the n-groupoid (Q, A) [: 2.2.1, 2.2.2, 2.2.4], we conclude that the following equalities hold

$$A(A((a_1^{n-2}, a)^{-1_1}, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1_2}) = (a_1^{n-2}, a)^{-1_2} \text{ and}$$
$$A(A((a_1^{n-2}, a)^{-1_1}, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1_2}) = (a_1^{n-2}, a)^{-1_1},$$

i. e., that $^{-1_2} = ^{-1_1}$.

3.3. Proposition: *Let $n \geq 2$ and let (Q, A) be an n-groupoid. Further on, let the $(1, n)$-associative law holds in (Q, A) [:2.1.1] and let for every $a_1^n \in Q$ there is* **at least one** *$x \in Q$ and* **at least one** *$y \in Q$ such that the following equalities $A(a_1^{n-1}, x) = a_n$ and $A(y, a_2^n) = a_n$ hold. Then, the following statement hold*

(I) (Q, A) has an $\{1, n\}$-neutral operation **e** *(:2.2.1, 2.2.4);*

(II) The $(2n-1)$*-groupoid* $(Q, \overset{2}{A})$, *where* $\overset{2}{A}(x_1^{2n-1}) \overset{def}{=} A(A(x_1^n), x_{n+1}^{2n-1})$, *has an* $\{1, 2n-1\}$*-neutral operation* E; *and*

(III) If

$$(a_1^{n-2}, a)^{-1} \overset{def}{=} \mathsf{E}(a_1^{n-2}, a, a_1^{n-2}),$$

then the laws (2_L)*-*(4_L) *and* (2_R)*-*(4_R) *from 2.3 hold in the algebra* $(Q, \{A, ^{-1}, \mathrm{e}\})$ *[of the type* $(n, n-1, n-2)$*]*.[3]

Proof. 1) The proof of statement (I):

Let a be an arbitrary (fixed) element of the set Q. Then for every sequence a_1^{n-2} over Q [: $n \geq 2$; 1] there is **at least one** $\mathrm{e}_L(a_1^{n-2}) \in Q$ [: $y = \mathrm{e}_L(a_1^{n-2})$] such that the following equality holds

$$A(\mathrm{e}_L(a_1^{n-2}), a_1^{n-2}, a) = a. \tag{a}$$

On the other hand, for every $b \in Q$ and for every sequence k_1^{n-2} over Q there is **at least one** $k \in Q$ [: $x = k$] such that the following equality holds

$$b = A(a, k_1^{n-2}, k). \tag{b}$$

Further on, by (a), (b) and the assumption that the $(1, n)$-associate law holds in (Q, A), we conclude that the following series equalities hold:

$$A(\mathrm{e}_L(a_1^{n-2}), a_1^{n-2}, b) = A(\mathrm{e}_L(a_1^{n-2}), a_1^{n-2}, A(a, k_1^{n-2}, k)) =$$
$$A(A(\mathrm{e}_L(a_1^{n-2}), a_1^{n-2}, a), k_1^{n-2}, k) = A(a, k_1^{n-2}, k) = b,$$

whence we conclude that for every $b \in Q$ and for every sequence a_1^{n-2} over Q [: $n \geq 2$; 1] the following equality holds

$$A(\mathrm{e}_L(a_1^{n-2}), a_1^{n-2}, b) = b,$$

i. e., that (Q, A) has [at least one] **left** $\{1, n\}$-neutral operation e_L. Similarly, it is possible to prove that there is a **right** $\{1, n\}$-neutral operation e_R in (Q, A). Finally, by Proposition 2.2.3, we conclude that there is an $\{1, n\}$-neutral operation $\mathbf{e}$ [$= \mathrm{e}_L = \mathrm{e}_R$].

[3] For $n = 2$ (Q, A) is a group, $\mathrm{e}(a_1^{n-2})$ [$= \mathrm{e}(\emptyset)$; 1] is its neutral element and $\mathsf{E}(a)$ is the inverse element of a with respect to $\mathrm{e}(\emptyset)$. For $n \geq 2$ there are algebras $(Q, \{A, ^{-1}, \mathrm{e}\})$ in which laws (2_L)-(4_L), (2_R)-(4_R) hold, and algebra (Q, A) is an $(1, n)$-associative n-groupoid but (Q, A) is not an n-group.

2) The proof of (II):

The $(1, 2n-1)$-associative law holds in $(Q, \overset{2}{A})$, where

$$\overset{2}{A}(x_1^{2n-1}) \overset{def}{=} A(A(x_1^n), x_1^{2n-1}),$$

and for every $a_1^{2n-1} \in Q$ there is **at least one** $x \in Q$ and **at least one** $y \in Q$ such that the following equalities hold $\overset{2}{A}(a_1^{2n-2}, x) = a_{2n-1}$ and $\overset{2}{A}(y, a_1^{2n-2}) = a_{2n-1}$. By (I), whence we conclude that the $(2n-1)$-groupoid $(Q, \overset{2}{A})$ has an $\{1, 2n-1\}$-neutral operation [let it be denoted by] E.

3) The proof of the statement (III):

Since E is an $\{1, 2n-1\}$-neutral operation of the $(2n-1)$-groupoid $(Q, \overset{2}{A})$, by the definitions of the operations $\overset{2}{A}$ and $^{-1}$, and by the assumption that the $(1, n)$-associative law holds in (Q, A), we conclude that for every $a, x \in Q$ and for every sequence a_1^{n-2} over Q [: $n \geq 2$; 1] the following equalities hold

$$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, x)) = x \text{ and}$$
$$A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = x,$$

whence, by the substitution $x = \mathrm{e}(a_1^{n-2})$, where e is an $\{1, n\}$-neutral operation of the n-groupoid (Q, A) [:(I)], we conclude that the following equalities hold

$$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathrm{e}(a_1^{n-2}) \text{ and}$$
$$A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathrm{e}(a_1^{n-2}).$$

□

By the definition of the n-group [:2.1.1] and by proposition 3.3, we conclude that the following proposition holds:

3.4. Proposition: *Let* $n \geq 2$ *and let* (Q, A) *be an* n*-group. Then, the following statements hold:*

(I) (Q, A) *has an* $\{1, n\}$*-neutral operation* e[4]

(II) $(Q, \overset{2}{A})$, *where* $\overset{2}{A}(x_1^{2n-1}) \overset{def}{=} A(A(x_1^n), x_{n+1}^{2n-1})$ *has an* $\{1, 2n-1\}$*-neutral operation* E; *and*

[4] [10]. There are n-groups without $\{i, j\} \neq \{1, n\}$ [:[13]]. In [13] n-groups with $\{i, j\}$-neutral operations for $\{i, j\} \neq \{1, n\}$ are described.

(III) If

$$(a_1^{n-2}, a)^{-1} \stackrel{def}{=} E(a_1^{n-2}, a, a_1^{n-2}),$$

then in algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $\langle n, n-1, n-2\rangle$ the laws $(1_L) - (4_L)$ and $(1_R) - (4_R)$ from 2.3 hold. [5]

In the present paper we use the following proposition:

3.5. Proposition: *Let $n \geq 3$ and let (Q, A) be an n-groupoid. Then the following statements hold:*

(I) If

(a) the $\langle 1, 2\rangle$-associative law holds in (Q, A), and

(b) for every $x, y, a_1^{n-1} \in Q$ the following implication holds

$$A(x, a_1^{n-1}) = A(y, a_1^{n-1}) \Rightarrow x = y,$$

then (Q, A) is an n-semigroup; and

(II) If

($\bar{a}$) the $\langle n-1, n\rangle$-associative law holds in (Q, A), and

($\bar{b}$) for every $x, y, a_1^{n-1} \in Q$ the following implication holds

$$A(a_1^{n-1}, x) = A(a_1^{n-1}, y) \Rightarrow x = y,$$

then (Q, A) is an n-semigroup.

Proof. In the proof of the proposition we use the method of E. I. Sokolov from [9] [from the proof of Theorem 1 in [9]].

Let $n \geq 3$ and let the $\langle 1, 2\rangle$-associative law holds in the n-groupoid (Q, A). In addition, suppose that the $\langle i, i+1\rangle$-associative law, $i \in \{1, \ldots, n-2\}$, holds in the n-groupoid (Q, A). Then, by (a) and (b) [from (I)], we conclude

[5] In [11] the n-group, $n \geq 2$, is described as a variety $(Q, \{A, ^{-1}\})$ of the type $\langle n, n-1\rangle$. A particular comment on that variety is given in section 5 [of the present paper].

that for every $a_1^{2n-1}, b_1^{n-1} \in Q$ the following series of implications hold

$$\begin{aligned}
&A(a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{2n-1}) = A(a_1^i, A(a_{i+1}^{i+n}), a_{i+n+1}^{2n-1}) \Rightarrow\\
&A(b_1, A(a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{2n-1}), b_2^{n-1}) =\\
&A(b_1, A(a_1^i, A(a_{i+1}^{i+n}), a_{i+n+1}^{2n-1}), b_2^{n-1}) \Rightarrow\\
&A(A(b_1, a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{2n-2}), a_{2n-1}, b_2^{n-1}) =\\
&A(A(b_1, a_1^i, A(a_{i+1}^{i+n}), a_{i+n+1}^{2n-2}), a_{2n-1}, b_2^{n-1}) \Rightarrow\\
&A(b_1, a_1^{i-1}, A(a_i^{i+n-1}), a_{i+n}^{2n-2}) = A(b_1, a_1^i, A(a_{i+1}^{i+n}), a_{i+n+1}^{2n-1}), a_{2n-2})
\end{aligned}$$

whence we conclude that then, in (Q, A) the $\langle i+1, i+2\rangle$-associate law also holds. Thus, (Q, A) is an n-semigroup.

Similarly it is possible to prove that the statement (II) also holds.

4. Main results

4.1. Theorem: *Let $n \geq 2$ and let (Q, A) be an n-groupoid. Then, the following statements hold:*

(a_1) (Q, A) is an n-group iff there are mappings $^{-1}$ and $\mathbf{e}$, respectively, of the sets Q^{n-1} and Q^{n-2} into the set Q such that the laws (1_R), (2_L) and (3_L) from 2.3 [Tabl. 1] hold in algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $\langle n, n-1, n-2\rangle$;

(b_1) There is at most one pair of mappings $^{-1}$ and $\mathbf{e}$, respectively of the sets Q^{n-1} and Q^{n-2} into the set Q such that the laws (1_R), (2_L) and (3_L) from 2.3 [Tabl. 1] hold in algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $\langle n, n-1, n-2\rangle$; and

(c_1) The laws (1_R), (2_L) and (3_L) from 2.3 [Tabl. 1] are independent.

Proof. 1) Proof of the statement (a_1):

1) $\Rightarrow$:

Let (Q, A) be an n-group. Then, by Proposition 3.4, there are an algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $\langle n, n-1, n-2\rangle$ in which the laws (1_R), (2_L) and (3_L) from 2.3 [Tabl. 1] hold.

1) $\Leftarrow$:

Let $(Q, \{A, ^{-1}, \mathbf{e}\})$ be a variety of the type $\langle n, n-1, n-2\rangle$ in which the laws (1_R), (2_L) and (3_L) from 2.3 [Tabl. 1] hold. We prove respectively that in that case the following statements hold:

$1^\circ (\forall a_i \in Q)_1^{n-1}(\forall x \in Q)(\forall y \in Q)(A(a_1^{n-1}, x) = A(a_1^{n-1}, y) \Rightarrow x = y)$;

$2^\circ (Q, A)$ is an n-semigroup;

$3^\circ (\forall a_i \in Q)_1^{n-2}(\forall x \in Q) A(x, a_1^{n-2} \mathrm{e}(a_1^{n-2})) = x$;

$4^\circ (\forall a_i \in Q)_1^{n-2}(\forall a \in Q) A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = \mathrm{e}(a_1^{n-2})$;

$5^\circ (\forall a_i \in Q)_1^{n-1}(\forall x \in Q)(\forall y \in Q)(A(x, a_1^{n-1}) = A(y, a_1^{n-1}) \Rightarrow x = y)$; and

$6^\circ (Q, A)$ is an n-quasigroup.

By 2° and 6°, by the definition of a n-group [:2.1.1], we conclude that (Q, A) is an n-group.

The proof of the statement 1°:

The case $n = 2$: since by the assumption the laws $(1_R)^6$, (2_L) and (3_L) hold in $(Q, \{A, ^{-1}, \mathrm{e}\})$, we have that for every $a, x, y \in Q$ [:$a_1^{n-2} = \emptyset$; 1] the following series of implications hold:

$$A(a, x) = A(a, y) \Rightarrow A(a^{-1}, A(a, x)) = A(a^{-1}, A(a, y)) \Rightarrow$$
$$A(A(a^{-1}, a), x) = A(A(a^{-1}, a), y) \Rightarrow A(\mathrm{e}(\emptyset), x) = A(\mathrm{e}(\emptyset)), y) \Rightarrow$$
$$x = y[: 2.2.3, 2.3].$$

The case $n > 2$: since by the assumption the laws (1_R) and (2_L) hold in $(Q, \{A, ^{-1}, \mathrm{e}\})$, we have that for every $a, x, y \in Q$, for every sequence a_1^{n-2} over Q and for every sequence c_1^{n-3} over Q [:$n \geq 3$; 1] the following series of implications holds:

$$A(a_1^{n-2}, a, x) = A(a_1^{n-2}, a, y) \Rightarrow$$
$$A(\mathrm{e}(c_1^{n-3}, a), c_1^{n-3}, \mathrm{e}(a_1^{n-2}), A(a_1^{n-2}, a, x)) =$$
$$A(\mathrm{e}(c_1^{n-3}, a), c_1^{n-3}, \mathrm{e}(a_1^{n-2}), A(a_1^{n-2}, a, y)) \Rightarrow$$
$$A(\mathrm{e}(c_1^{n-3}, a), c_1^{n-3}, A(\mathrm{e}(a_1^{n-2}), a_1^{n-2}, a), x) =$$
$$A(\mathrm{e}(c_1^{n-3}, a), c_1^{n-3}, A(\mathrm{e}(a_1^{n-2}), a_1^{n-2}, a), y) \Rightarrow$$
$$A(\mathrm{e}(c_1^{n-3}, a), c_1^{n-3}, a, x) = A(\mathrm{e}(c_1^{n-3}, a), c_1^{n-3}, a, y) \Rightarrow$$
$$x = y.$$

[6] (1_R): $A(A(x_1^2), x_3) = A(x_1, A(x_2^3))$.

The proof of the statement 2°:

For $n = 2$ the statement 2° is an immediate consequence of the definition of a semigroup (2-semigroup) and of the assumption that the law (1_R) holds in (Q, A). For $n > 2$ the statement 2° holds by the assumption that the law (1_R) holds in (Q, A), by statement 1° and by Proposition 3.5.

Proof of the statement 3°:

Since the $\langle 1, n\rangle$-associative law holds in (Q, A) [:2°], by the assumption that the laws (3_L) and (2_L) as well as 1° hold in $(Q, \{A, ^{-1}, \mathrm{e}\})$, we conclude that for every $a, b \in Q$ and for every sequence a_1^{n-2} over Q [:$n \geq 2$; 1] the following sequence of implications holds:

$$A(a, a_1^{n-2}, \mathrm{e}(a_1^{n-2})) = b \Rightarrow$$
$$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, \mathrm{e}(a_1^{n-2}))) =$$
$$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \Rightarrow$$
$$A(A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a), a_1^{n-2}, \mathrm{e}(a_1^{n-2})) =$$
$$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \Rightarrow$$
$$A(\mathrm{e}(a_1^{n-2}), a_1^{n-2}, \mathrm{e}(a_1^{n-2})) = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \Rightarrow$$
$$\mathrm{e}(a_1^{n-2}) = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \Rightarrow$$
$$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \Rightarrow$$
$$a = b.$$

The proof of the statement 4°:

Since the $\langle 1, n\rangle$-associative law [:2°] holds in (Q, A), by the assumption that laws (3_L) and (2_L) as well as the statements 3° and 1° hold in $(Q, \{A, ^{-1}, \mathrm{e}\})$, we conclude that for every $a, b \in Q$ and for every sequence a_1^{n-2} over Q

$[:n \geq 2;\ 1]$ the following sequence of implications holds:

$$\begin{aligned}
&A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = b \Rightarrow\\
&A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1})) =\\
&A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \Rightarrow\\
&A(A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) =\\
&A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \Rightarrow\\
&A(\mathrm{e}(a_1^{n-2}), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \Rightarrow\\
&A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, \mathrm{e}(a_1^{n-2})) = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \Rightarrow\\
&\mathrm{e}(a_1^{n-2}) = b.
\end{aligned}$$

The proof of the statement 5°:

Since the $\langle 1, n\rangle$-associative law $[:2^\circ]$ as well as the statements 4° and 3° hold in (Q, A), we conclude that for every $x, y, a \in Q$ and for every sequence a_1^{n-2} over Q $[:n \geq 2;\ 1]$ the following series of implications holds:

$$\begin{aligned}
&A(x, a_1^{n-2}, a) = A(y, a_1^{n-2}, a) \Rightarrow\\
&A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = A(A(y, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \Rightarrow\\
&A(x, a_1^{n-2}, A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1})) = A(y, a_1^{n-2}, A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1})) \Rightarrow\\
&A(x, a_1^{n-2}, \mathrm{e}(a_1^{n-2})) = A(y, a_1^{n-2}, \mathrm{e}(a_1^{n-2})) \Rightarrow\\
&x = y.
\end{aligned}$$

The proof of the statement 6°:

By 5° and 1° and the corresponding monotones we conclude that the following formulas hold:

(l) $(\forall a_i \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)(\forall y \in Q)(A(x, a_1^{n-2}, a) = A(y, a_1^{n-2}, a) \Leftrightarrow x = y)$; and

(r) $(\forall a_i \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)(\forall y \in Q)(A(a, a_1^{n-2}, x) = A(a, a_1^{n-2}, y) \Leftrightarrow x = y)$.

6_1°: For arbitrary $a, b \in Q$ and an arbitrary sequence a_1^{n-2} over Q $[:n \geq 2;\ 1]$ the equation

$$A(x, a_1^{n-2}, a) = b$$

over the unknown x has exactly one solution which can be expressed in the following way

$$A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}).$$

Indeed, by (l), 2°, 4° and 3°, we conclude that for every $x, a, b \in Q$ and for every sequence a_1^{n-2} over Q the following series of equivalences holds

$$\begin{aligned}
&A(x, a_1^{n-2}, a) = b \Leftrightarrow\\
&A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \Leftrightarrow\\
&A(x, a_1^{n-2}, A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1})) = A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \Leftrightarrow\\
&A(x, a_1^{n-2}, \mathrm{e}(a_1^{n-2})) = A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \Leftrightarrow\\
&x = A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}).^{7}
\end{aligned}$$

Similarly, by (r), (2°) and by the assumption that the laws (3_L) and (2_L) hold in $(Q, \{A, {}^{-1}, \mathrm{e}\})$, we conclude that the following proposition holds:

6_2°: For arbitrary $a, b \in Q$ and for an arbitrary sequence a_1^{n-2} over Q $[:n \geq 2;\ 1]$ the equation

$$A(a, a_1^{n-2}, x) = b$$

over the unknown x has exactly one solution which can be expressed in the following way

$$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b).^{8}$$

6_3°: Let $n \geq 3$, $i \in \{1, \ldots, n-2\}$, let a_1^{n-3} be a sequence over Q $[:n \geq 3;1]$ and let $a, b, c \in Q$. Then, the equation

$$A(a, a_1^{i-1}, x, a_i^{n-3}, b) = c$$

over the unknown x has at most one solution in Q

Indeed, by 2°, 6_1° and 6_2°, we conclude that for every $a, b, x, y \in Q$, for every sequence a_1^{n-3} over Q $[:n \geq 3;1]$ and for every sequence c_1^{n-1} over Q the

[7] For $n = 2$ the following holds: $A(x, a) = b \Leftrightarrow x = A(b, a^{-1})$.
[8] For $n = 2$ the following hold: $A(a, x) = b \Leftrightarrow x = A(a^{-1}, b)$.

following series of implications holds:

$$A(a, a_1^{i-1}, x, a_i^{n-3}, b) = A(a, a_1^{i-1}, y, a_i^{n-3}, b) \Rightarrow$$
$$A(c_{i+1}^{n-1}, A(a, a_1^{i-1}, x, a_i^{n-3}, b), c_1^i) = A(c_{i+1}^{n-1}, A(a, a_1^{i-1}, y, a_i^{n-3}, b), c_1^i) \Rightarrow$$
$$A(A(c_{i+1}^{n-1}, a, a_1^{i-1}, x), a_i^{n-3}, b, c_1^i) = A(A(c_{i+1}^{n-1}, a, a_1^{i-1}, y), a_i^{n-3}, b, c_1^i) \Rightarrow$$
$$A(c_{i+1}^{n-1}, a, a_1^{i-1}, x) = A(c_{i+1}^{n-1}, a, a_1^{i-1}, y) \Rightarrow$$
$$x = y.$$

6_4°: Let $n \geq 3$, $i \in \{1, \dots, n-2\}$, let a_1^{n-3} be a sequence over Q [$:n \geq 3;1$] and let $a, b, c \in Q$. Then, the equation

$$A(a, a_1^{i-1}, x, a_i^{n-3}, b) = c$$

over the unknown x has at least one solution in Q.

Indeed, by the proof of statement 6_3° and by the statement 2°, we conclude that for every sequence a_1^{n-3} over Q [$:n \geq 3;1$] and for every sequence c_1^{n-1} over Q the following equivalence holds:

$$A(a, a_1^{i-1}, x, a_i^{n-3}, b) = c \Leftrightarrow$$
$$A(A(c_{i+1}^{n-1}, a, a_1^{i-1}, x), a_i^{n-3}, b, c_1^i) = A(c_{i+1}^{n-1}, c, c_1^i),$$

whence, by 6_1° and 6_2°, we conclude 6_4° holds.

2) Proof of the statement (b_1):

Let $n \geq 2$ and let (Q, A) be an n-groupoid. In addition, assume that there are mappings

$$^{-1_k} : Q^{n-1} \to Q \text{ and } \mathbf{e}_k : Q^{n-2} \to Q,\ k \in \{1, 2\},$$

such that the laws (1_R), (2_L) and (3_L) hold in the algebras $(Q, \{A, ^{-1_1}, \mathbf{e}_1\})$ and $(Q, \{A, ^{-1_2}, \mathbf{e}_2\})$. Then: (i) by 3° from the proof of the statement (a_1), by 2.2.1 and 2.2.4, we conclude that $\mathbf{e}_1 = \mathbf{e}_2$; and (ii) since $\mathbf{e}_1 = \mathbf{e}_2$ [:(i)], by 4° from the proof of the statement (a_1) and by Proposition 3.2, we conclude that $^{-1_1} = ^{-1_2}$.

3) Proof of the statement (c_1):

3_1) The laws (1_R) and (2_L) hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $\langle n, n-1, n-2\rangle$, where $n \geq 2$, $|Q| > 1$, $A(x_1^n) \stackrel{def}{=} x_1$, $^{-1}$ an arbitrary $(n-1)$-ary operation in Q and $\mathbf{e}(a_1^{n-2}) \stackrel{def}{=} c$ - constant. However, the law (3_L) does not hold.

3_2) The laws (1_R) and (3_L) hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $\langle n, n-1, n-2\rangle$, where $n \geq 2$, $|Q| > 1$, $A(x_1^n) \stackrel{def}{=} x_1$, $\mathbf{e}(a_1^{n-2}) \stackrel{def}{=} c$ - constant and $(a_1^{n-2}, a)^{-1} \stackrel{def}{=} \mathbf{e}(a_1^{n-2})$. However, the law (2_L) does not hold.

3_3) **The case** $n > 2$: Let $(Q, \square)$ be a group, $^{-1}$ its inversing operation, and let (Q, B) be an $(n-2)$-groupoid which is not an $(n-2)$-quasigroup [for $n = 3$: $B \notin Q$!]. Then (Q, A) satisfies conditions of Proposition 3.3, where

$$A(x, a_1^{n-2}, y) \stackrel{def}{=} x \,\square\, (B(a_1^{n-2}))^{-1} \,\square\, y.$$

Thus, there is an algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $\langle n, n-1, n-2\rangle$, in which the laws (2_L) and (3_L) hold. However, the law (1_R) fails to hold in $(Q, \{A, ^{-1}, \mathbf{e}\})$. Indeed, if the law (1_R) holds in $(Q, \{A, ^{-1}, \mathbf{e}\})$, then by the statement (a_1) [:1)$\Leftarrow$] (Q, A) is an n-group, which contradict the assumption that (Q, B) is not an $(n-2)$-quasigroup.

The case $n = 2$: Let (Q, A) be a Moufang loop which is not a group, let $e = \mathbf{e}(\emptyset)$ be its neutral element and $^{-1}$ its inversing operation [:[2-3]]. Then the laws (2_L) and (3_L) hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $\langle 2, 1, 0\rangle$. However, the law (1_R) [$=(1_L)$] does not hold.

□

All results of this part [including Theorem 4.1] can be described by means of Tables 1-6 in the following way:

4.2. **Theorem:** *Let $n \geq 2$ and let (Q, A) be an n-groupoid. Then the following statements hold:*

(a_i) (Q, A) is an n-group iff **there are** *mappings $^{-1}$ and* $\mathbf{e}$, *respectively, of the sets Q^{n-1} and Q^{n-2} into the set Q such that in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $\langle n, n-1, n-2\rangle$ the laws from the marked fields from the Table* i[9] *hold;*

(b_i) there is at most one pair of mappings $^{-1}$ and $\mathbf{e}$, *respectively, of the sets Q^{n-1} and Q^{n-2} into the set Q such that in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the*

[9] $i \in \{1, \dots, 6\}$

type $\langle n, n-1, n-2\rangle$ *the laws from the marked fields from the Table* i *hold; and*

(c_i) *the laws from the marked fields of the Table* i *are mutually independent.*

Proof. The proof of the statement (a_i) [$i \in \{2, \ldots, 6\}$[10]]:

1) $\Rightarrow$:

Let (Q, A) be an n-group. Then (a_i) [for every $i \in \{1, \ldots, 6\}$] in the direction $\Rightarrow$ holds by Proposition 3.4.

1) $\Leftarrow$:

$i = 2$: Let the laws (1_L), (2_R) and (3_R) from 2.3 [Tab. 2] hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $\langle n, n-1, n-2\rangle$. Then the following statements hold:

°1 $(\forall a_i \in Q)_1^{n-1} (\forall x \in Q)(\forall y \in Q)(A(x, a_1^{n-1}) = A(y, a_1^{n-1}) \Rightarrow x = y)$;

°2 (Q, A) is an n-semigroup [in (Q, A) holds the law (1_R) from 2.3];

°3 $(\forall a_i \in Q)_1^{n-2} (\forall x \in Q)(A(\mathbf{e}(a_1^{n-2}), a_1^{n-2}, x) = x$; and

°4 $(\forall a_i \in Q)_1^{n-2} (\forall x \in Q)(A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = \mathbf{e}(a_1^{n-2})$.

The statements °1-°4 could be proved imitating the proofs of statements 1°-4° [:proof of the Theorem 4.1]. Since from °2-°4 it follows that the laws (1_R), (2_L) and (3_L) from 2.3 [Tab. 1] hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$, we conclude by the corresponding part of Theorem 4.1, that (Q, A) is an n-group.

$i = 3$: Let the laws (1_L), (2_L) and (4_R) from 2.3 [Tab. 3] hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$. Since the law (4_R) holds in $(Q, \{A, ^{-1}, \mathbf{e}\})$, we conclude that for every $x, y, a \in Q$ and for every sequence a_1^{n-2} over Q [:$n \geq 2$; 1] the following implication holds

$$A(x, a_1^{n-2}, a) = A(y, a_1^{n-2}, a) \Rightarrow x = y.$$

In addition, if we take into account that the law (1_L) holds in $(Q, \{A, ^{-1}, \mathbf{e}\})$, then by Proposition 3.5, we conclude that for $n \geq 3$ the $\langle 1, n\rangle$-associative law holds in (Q, A). For $n = 2$ $(1_L) = (1_R)$ is the $\langle 1, n\rangle$-associative law. On the other hand, if in (4_R) we put $x = \mathbf{e}(a_1^{n-2})$, and take into consideration (2_L), we conclude that also the law (3_R) holds in $(Q, \{A, ^{-1}, \mathbf{e}\})$. Finally,

[10] $i = 1$: Theorem 4.1.

since the laws (4_R), $A(A(x_1^n), x_{n+1}^{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))$ and (3_R) hold in $(Q, \{A, ^{-1}, \mathbf{e}\})$, we conclude that the law (2_R) holds in $(Q, \{A, ^{-1}, \mathbf{e}\})$. By ($a_i$) for $i = 2$ [$\Leftarrow$], (Q, A) is thus an n-group.

$i = 4$: let the laws (1_R), (2_L) and (4_R) from 2.3 [Tab. 4] hold in the algebra $(Q, \{A, ^{-1}, \mathbf{e}\})$ of the type $\langle n, n-1, n-2\rangle$. For $n = 2$ the case "$i = 4$" coincides with the case "$i = 3$". Let $n \geq 3$. Then, by (1_R) and (2_L), and also by Proposition 3.5, as in the proof of the Theorem 4.1 [:proofs of the statements 1° and 2°], we conclude that also the law (1_L) holds in $(Q, \{a, ^{-1}, \mathbf{e}\})$. In that way, the case "$i = 4$" is transferred to the case "$i = 3$", and therefore (Q, A) is an n-group [also for $n \geq 3$].

The case "$i = 3$" is transferred to the case "$i = 2$". Similarly, the case "$i = 5$" could be transferred to the case "$i = 1$" [:Theorem 4.1]. The case "$i = 4$" was transferred to the case "$i = 3$". Similarly, the case "$i = 6$" can be transferred to the case "$i = 5$". [The case "$i = 2$" was transferred to the case "$i = 1$".]

2) The proof of the statement (b_i) [$i \in \{2, \ldots, 6\}$]

Let $n \geq 2$ and let be an n-groupoid. In addition, suppose that there are mappings

$$^{-1_k} : Q^{n-1} \to Q \text{ and } \mathbf{e}_k : Q^{n-2} \to Q,\ k \in \{1, 2\},$$

such that the laws from the marked fields of the Table i [$i \in \{2, \ldots, 6\}$; 2.3] hold in the algebras $(Q, \{A, ^{-1_1}, \mathbf{e}_1\})$ and $(Q, \{A, ^{-1_2}, \mathbf{e}_2\})$.

For every $i \in \{2, \ldots, 6\}$ in the algebras $(Q, \{A, ^{-1_1}, \mathbf{e}_1\})$ and $(Q, \{A, ^{-1_2}, \mathbf{e}_2\})$ either the law (2_L) or the law (2_R) [from 2.3] holds. It means that $\mathbf{e}_1$ and $\mathbf{e}_2$ are either left $\{1, n\}$-neutral operations of the n-groupoid (Q, A) or right $\{1, n\}$-neutral operations of (Q, A). In addition, by (a_i) [$\Leftarrow$], (Q, A) is an n-group for every $i \in \{2, \ldots, 6\}$, whence, by Proposition 3.4, we conclude that (Q, A) has an $\{1, n\}$-neutral operation since $\mathbf{e}$ is a left $\{1, n\}$-neutral operation of the n-groupoid (Q, A) and since $\mathbf{e}$ is a right $\{1, n\}$-neutral operation of the n-groupoid (Q, A), by Proposition 2.2.3, we conclude that $\mathbf{e}_1 = \mathbf{e}$ and $\mathbf{e}_2 = \mathbf{e}$ i.e., that $\mathbf{e}_1 = \mathbf{e}_2$.

For $i = 2$, since $\mathbf{e}_1 = \mathbf{e}_2$, for every $a \in Q$, and for every sequence a_1^{n-2} over Q the following equality holds

$$A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1_1}) = A(a, a_1^{n-2}, (a_1^{n-2}, a)^{-1_2}),$$

whence, since (Q, A) is an n-group [:(a_i), $\Leftarrow$], we conclude that $^{-1_1} = ^{-1_2}$ [:2.1.1]. In the cases $i \in \{3,4\}$, for every $a, x \in Q$ and for every sequence a_1^{n-2} over Q the following equality holds

$$A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1_1}) = A(A(x, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1_2})$$

and is the cases $i \in \{5,6\}$ holds the equality

$$A((a_1^{n-2}, a)^{-1_1}, a_1^{n-2}, A(a, a_1^{n-2}, x)) = A((a_1^{n-2}, a)^{-1_2}, a_1^{n-2}, A(a, a_1^{n-2}, x)),$$

whence, since (Q, A) is an n-group, we conclude that for every $i \in \{3,4,5,6\}$, $^{-1_1} = ^{-1_2}$.

3) The proof of the statement (c_i) [$i \in \{2, \ldots, 6\}$]:

By means of operations A, $^{-1}$ and e in the set Q [$|Q| > 1$] which are defined in the proof of the statement (c_1) [:proof of Theorem 4.1] for every $i \in \{2, \ldots, 6\}$ one could define algebras of the type $\langle n, n-1, n-2 \rangle$ such that exactly two of the laws in the marked fields of the Table i hold and that the third law does not hold. (In Moufang's loops (Q, A), also the following laws hold: $A(a^{-1}, A(a, x)) = x$ and $A(A(a, x), a^{-1}) = x$ [:[2-3]].)

5. On n-groups, $n \geq 2$, as varieties of type $\langle n, n-1 \rangle$

Useful role in the **description** of main results of the present article [:4.1, 4.2] play Tables 1-6 [:2.3]. In this part using similarly Tables 7 and 8 we **describe** two varieties of the type $\langle n, n-1 \rangle$ which represent characterizations of n-groups for $n \geq 2$. Besides, the main part of the announced result is already known: Remark 5.2.

	L	R
1		•
2		
3		
4	•	•

Tabl. 7

	L	R
1	•	
2		
3		
4	•	•

Tabl. 8

5.1. Theorem: *Let $n \geq 2$ and let (Q, A) be an n-groupoid. Then the following statements hold:*

(a_i) *(Q, A) is an n-group iff* **there is** *a mapping $^{-1}$ of the set Q^{n-1} into the set Q such that in the algebra $(Q, \{A, ^{-1}\})$ of the type $\langle n, n-1 \rangle$ the laws from the marked fields of the Table i [$i \in \{7,8\}$] hold;*

(b_i) *There is at most one $(n-1)$-ary operation $^{-1}$ in the set Q such that in the algebra $(Q, \{A, ^{-1}\})$ of the type $\langle n, n-1 \rangle$ the laws from the marked fields of the Table i [$i \in \{7,8\}$] hold; and*

(c_i) *The laws from the marked fields of the Table i [$i \in \{7,8\}$] are mutually independent.*

Proof. The proof is based on the part of the from Part 3 of Theorem 4.1.

1) The proof of the statement (a_i) [$i \in 7,8\}$]:

1) $\Rightarrow$:

Let (Q, A) be an n-group, $n \geq 2$. Then, by Proposition 3.4, the statement (a_i) in the direction "$\Rightarrow$" holds for $i = 7$ as well as for $i = 8$.

1) $\Leftarrow$:

In the variety $(Q, \{A, ^{-1}\})$ described by Tab. 7 as well as in the variety $(Q, \{A, ^{-1}\})$ describe by the Table 8 [$i \in \{7,8\}$] the laws (4_L) and $4_R)$ hold. Therefore, we conclude that for $i = 7$ as well as for $i = 8$ the following formula hold

(k_r)
$$(\forall a_j \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)(\forall y \in Q)$$
$$(A(a, a_1^{n-2}, x) = A(a, a_1^{n-2}, y) \Rightarrow x = y) \text{ and}$$

(k_l)
$$(\forall a_j \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)(\forall y \in Q)$$
$$(A(x, a_1^{n-2}, a) = A(y, a_1^{n-2}, a) \Rightarrow x = y),$$

whence, by the corresponding monotonies, we conclude that also the following formulas hold

(k_r)
$$(\forall a_j \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)(\forall y \in Q)$$
$$(A(a, a_1^{n-2}, x) = A(a, a_1^{n-2}, y) \Leftrightarrow x = y) \text{ and}$$

(k_l)
$$(\forall a_j \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)(\forall y \in Q)$$
$$(A(x, a_1^{n-2}, a) = A(y, a_1^{n-2}, a) \Leftrightarrow x = y).$$

Since for $i = 7$ as well as for $i = 8$ the formulas (k_r) and (k_l) hold, by the laws (4_L) and (4_R), we conclude that for every $a, b, x, y \in Q$ and for every sequence a_1^{n-2} over Q the following two series of equivalences hold

$$A(a, a_1^{n-2}, x) = b \Leftrightarrow$$
$$A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, A(a, a_1^{n-2}, x)) = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \Leftrightarrow$$
$$x = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b) \text{ and}$$

$$A(y, a_1^{n-2}, a) = b \Leftrightarrow$$
$$A(A(y, a_1^{n-2}, a), a_1^{n-2}, (a_1^{n-2}, a)^{-1}) = A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}) \Leftrightarrow$$
$$y = A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1}),$$

whence, we conclude that the following two formulas hold

$$(e_r) \qquad \begin{array}{l} (\forall a_j \in Q)_1^{n-2} (\forall a \in Q)(\forall b \in Q)(\forall x \in Q) \\ (A(a, a_1^{n-2}, x) = b \Leftrightarrow x = A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, b)) \text{ and} \end{array}$$

$$(e_l) \qquad \begin{array}{l} (\forall a_j \in Q)_1^{n-2} (\forall a \in Q)(\forall b \in Q)(\forall x \in Q) \\ (A(y, a_1^{n-2}, a) = b \Leftrightarrow y = A(b, a_1^{n-2}, (a_1^{n-2}, a)^{-1})). \end{array}$$

Let $i = 7$. For $n = 2$ $(1_R) = (1_L)$ is the $\langle 1, n \rangle$-associative law. For $n > 2$, by (1_R) and (k_r), by Proposition 3.5, we conclude that $\langle 1, n \rangle$-associative law holds in (Q, A) [actually, that (Q, A) is an n-semigroup]. Hence, for every $n \geq 2$ the $\langle 1, n \rangle$-associative law holds in (Q, A). Therefore, and by the formulas (e_r) and (e_l), by Proposition 3.3, we conclude that there is an algebra $(Q, \{A, ^{-1_1}, \mathbf{e}\})$ in which the laws (2_L) and (3_L) hold. Hence since in (Q, A) [by the assumption (:Tabl. 7)] the $\langle n-1, n \rangle$-associative law holds, by Theorem 4.1, we conclude that (Q, A) is an n-group.

Similarly one could prove that (Q, A) is an n-group $(n \geq 2)$ also in the case $i = 8$ [:Tab. 8].

2) The proof of the statement (b_i) $[i \in \{7, 8\}]$:

Let $n \geq 2$ and let (Q, A) be an n-groupoid. In addition, assume that there are mappings $^{-1_1}$ and $^{-1_2}$ of the set Q^{n-1} into the set Q such that in the algebras $(Q, \{A, ^{-1_1}\})$ and $(Q, \{A, ^{-1_2}\})$ the laws from the marked fields of the Table i $[i \in \{7, 8\}]$ hold. Whence, by the statement (a_i) $[i \in \{7, 8\}; \Leftarrow]$, we conclude that (Q, A) is an n-group for $i = 7$ as well as for $i = 8$, whence, by Proposition 3.4 and by the definition of the n-group [:2.1.1], we conclude

that there is an $(n-1)$-ary operation $^{-1}$ in Q such that the laws (4_L) and (4_R) hold in the algebra $(Q, \{A, ^{-1}\})$, and that (Q, A) is a $\langle 1, n \rangle$-associative n-groupoid. Finally, by Proposition 3.1, we conclude that $^{-1_1} = ^{-1}$ and $^{-1_2} = ^{-1}$, i.e., that $^{-1_1} = ^{-1_2}$.

3) The proof of the statement (c_i) $[i \in \{7, 8\}]$:

See the proof of the statement (c_i) for $i \in \{2, 3, \ldots, 6\}$ [:proof of Theorem 4.2].

5.2. **Remark:** In [11], the author proved, among others, the following proposition: *Let $n \geq 2$ and let (Q, A) be an n-semigroup. Then, (Q, A) is an n-group iff there is a mapping $^{-1}$ of the set Q^{n-1} into the set Q such that the laws (4_L) and (4_R) from 2.3 hold in the algebra $(Q, \{A, ^{-1}\})$.* [For $n = 2$ it is a well known characterization of the group.] In [14] W. Dudek showed that for $n \geq 3$ the condition "...(Q, A) is an n-semigroup..." can be weakened to the condition "...(Q, A) is an $\langle 1, 2 \rangle$-associative n-groupoid..." or to the condition "...(Q, A) is an $\langle n-1, n \rangle$-associative n-groupoid...". [In [11] the author used the notation $f(a_1^{n-2}, a)$ instead of $(a_1^{n-2}, a)^{-1}$ and named f the inversing operation in the n-group (Q, A). The author use the same notation for example in the paper [12]. In [14] W. Dudek uses f instead of A, and $h(a_1^{n-2}, a)$ instead of $(a_1^{n-2}, a)^{-1}$.]

6. On two varieties of type $\langle n, n-2 \rangle$ characterizing n-groups for $n \geq 3$

In part 4 we describe six varieties of the type $\langle n, n-1, n-2 \rangle$ [each with three independent laws] by means of which we characterize n-groups for every $n \geq 2$. For $n = 2$ these are four [$:(1_L) = (1_R)$] known characterizations of a group. In addition, the operations $^{-1}$ and e represent, respectively, inversing and nullary [fixing a neutral element] operation in a group. In the description of n-groups, $n \geq 2$, as varieties of the type $\langle n, n-1 \rangle$ [section 5] the $\{1, n\}$-neutral operation [explicitly] does not exist. In the **description** of each of eight varieties characterizing n-groups for every $n \geq 2$, important role play Tables 1-8. In this section, by means of Tables 9 and 10 [in the fields of which are the same laws as in the fields of the Tables 1-8: laws $(1_L) - (4_L)$

and (1_R) – (4_R) from 2.3] similarly we describe two varieties of the type $\langle n, n-2\rangle$ giving characterizations of n-groups for every $n \geq 3$ but not for $n = 2$. Besides, the basic part of the announced result is already known: Remark 6.2.

	L	R
1		•
2	•	•
3		
4		

Tabl. 9

	L	R
1	•	
2	•	•
3		
4		

Tabl. 10

6.1. Theorem: *Let $n \geq 3$ and let (Q, A) be an n-groupoid. Then, the following statements hold:*

(a_i) *(Q, A) is an n-group iff* **there is** *a mapping* e *of the set Q^{n-2} into the set Q such that the laws from the marked fields of the Table* i *[i $\in \{9, 10\}$] hold in the algebra $(Q, \{A, \mathrm{e}\})$ of the type $\langle n, n-2\rangle$;*

(b_i) *There is at most one $(n-2)$-ary operation* e *in the set Q such that the laws from the marked fields of the Table* i *[i $\in \{9, 10\}$] hold in the algebra $(Q, \{A, \mathrm{e}\})$ of the type $\langle n, n-2\rangle$; and*

(c_i) *The laws from the marked fields of the Table* i *[i $\in \{9, 10\}$] are mutually independent.*

Proof. The proof is based on the part of the proof from section 3 and on Theorem 4.1.

1) The proof of the statement (a_i) [$i \in \{9, 10\}$]:

1) $\Rightarrow$:

Let (Q, A) be an n-group, $n \geq 3$. Then, by Proposition 3.4, the statement (a_i) in the direction "$\Rightarrow$" holds for $i = 9$ as well as for $i = 10$. [Besides, the statement (a_i), $i \in \{9, 10\}$, in the direction "$\Rightarrow$" holds also for $n = 2$.]

1) $\Leftarrow$:

If $n = 2$, then for $i = 9$ as well as for $i = 10$, (Q, A) is a semigroup with the neutral element $\mathrm{e}(\emptyset)$ [:2.2.2].

Let $n \geq 3$.

$i = 9$: Thus, let the laws (1_L), (2_L) and (2_R) from 2.3 [Tabl. 9] hold in the algebra $(Q, \{A, \mathrm{e}\})$ of the type $\langle n, n-2\rangle$.

By the assumption that the laws (1_R) and (2_L) hold in $(Q, \{A, \mathrm{e}\})$, we conclude that the following formula holds

$$(k_r) \qquad \begin{array}{l}(\forall a_j \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)(\forall y \in Q)\\ (A(a_1^{n-2}, a, x) = A(a_1^{n-2}, a, y) \Rightarrow x = y).\end{array}$$

Whence, by the corresponding monotony, we conclude that also the following formula holds

$$(k_r) \qquad \begin{array}{l}(\forall a_j \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)(\forall y \in Q)\\ (A(a_1^{n-2}, a, x) = A(a_1^{n-2}, a, y) \Leftrightarrow x = y).\end{array}$$

By (k_r) and by the assumption that the law (1_R) holds in (Q, A), using Proposition 3.5, we conclude that (Q, A) is an n-semigroup, whence it follows that the following statements hold

1 The law(1_L) holds in (Q, A); and

2 The $\langle 1, n\rangle$-associative law holds in (Q, A).

By the assumption that the law (2_R) holds in $(Q, \{A, \mathrm{e}\})$, and since the law (1_L) [:1] also holds in (Q, A), we conclude that the following formula holds

$$(k_l) \qquad \begin{array}{l}(\forall a_j \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)(\forall y \in Q)\\ (A(x, a, a_1^{n-2}) = A(y, a, a_1^{n-2}) \Rightarrow x = y).\end{array}$$

Whence, by the corresponding monotony, we conclude that the following formula holds

$$(k_l) \qquad \begin{array}{l}(\forall a_j \in Q)_1^{n-2}(\forall a \in Q)(\forall x \in Q)(\forall y \in Q)\\ (A(x, a, a_1^{n-2}) = A(y, a, a_1^{n-2}) \Leftrightarrow x = y).\end{array}$$

Since the laws (1_R), (2_L), (2_R) and (1_L) [:Tabl. 9, statement 1] and the statements (k_r) and (k_l) hold in $(Q, \{A, \mathrm{e}\})$, we conclude that for every $a, b, x \in Q$, for every sequence a_1^{n-2} over Q, and for every sequence c_1^{n-3} over Q [:$n \geq 3$; 1] the following two equivalences hold

$$A(a_1^{n-2}, a, x) = b \Leftrightarrow x = A(\mathrm{e}(c_1^{n-3}, a), c_1^{n-3}, \mathrm{e}(a_1^{n-2}), b) \text{ and}$$

$$A(x, a, a_1^{n-2}) = b \Leftrightarrow x = A(b, \mathrm{e}(a_1^{n-2}), c_1^{n-3}, \mathrm{e}(a, c_1^{n-3})),$$

whence, by the statement 2, using Proposition 3.3, we conclude that there are mappings $^{-1}$ and e_l, respectively, of the sets Q^{n-1} and Q^{n-2} into the set Q such that the laws (2_L) and (3_L) hold in the algebra $(Q, \{A, ^{-1_1}, \mathbf{e}_1\})$. Whence since the law (1_R) [:Tabl. 9] holds in (Q, A), by Theorem 4.1, we conclude that (Q, A) is an n-group.

$i = 10$: Thus, let the laws (1_L), (2_L) and (2_R) from 2.3 [Tabl. 10] hold in the algebra $(Q, \{A, \mathbf{e}\})$ of the type $(n, n-2)$.

By the assumption that the laws (1_L) and (2_R), hold in the algebra $(Q, \{A, \mathbf{e}\})$, we conclude that the following formula holds

$$(\forall a_j \in Q)(\forall a \in Q)(\forall x \in Q)(\forall y \in Q)$$
$$(A(x, a, a_1^{n-2}) = A(y, a, a_1^{n-2}) \Rightarrow x = y)$$

[:(k_l)]. Whence, since the law (1_L) holds in (Q, A), by Proposition 3.5, we conclude that also the law (1_R) holds in (Q, A). Thus: If the laws from the marked fields of the Table 10 hold in algebra $(Q, \{A, \mathbf{e}\})$, then also the laws from the marked fields of the Table 9 hold in $(Q, \{A, \mathbf{e}\})$. In that way, the case "$i = 10$" is transferred to the case "$i = 9$", whence we conclude that (Q, A) is an n-group.

2) The proof of the statement (b_i) [$i \in \{9, 10\}$]:

Using Tables 9 and 10, we conclude that $\mathbf{e}$ is a $\{1, n\}$-neutral operation of the n-groupoid (Q, A) [:2.2.1] for $i = 9$ as well as for $i = 10$. Whence, by Proposition 2.2.4, we conclude that statement (b_i) [:$i \in \{9, 10\}$] holds.

3) The proof of the statement (c_i) [$i \in \{9, 10\}$]:

See the proof of the statement (c_1) [:proof of Theorem 4.1].

6.2. Remark: In the paper [10] [where the notion of $\{i, j\}$-neutral operation of the n-groupoid is introduced: 2.2.1] also the following proposition was proved:

6.2.1. Proposition: *Let $n \geq 3$ and let (Q, A) be an n-semigroup. Then: (Q, A) is an n-group iff (Q, A) has a $\{1, n\}$-neutral operation.*

In Proposition 6.2.1, the condition "...(Q, A) is an n-semigroup..." can be weakened to the condition "...in an n-groupoid (Q, A) the law (1_L) from 2.3 holds..." or to the condition "...in an n-groupoid (Q, A) the law (1_R) from 2.3 holds..." [:Theorem 6.1 - (a_i), $i \in \{9, 10\}$].

The part of Theorem 1.4 from the paper [5] is the following proposition:

6.2.2. Proposition: *Let $n \geq 3$ and let (Q, A) be an n-semigroup. Then: (Q, A) is an n-group iff for every $a_1^{n-2} \in Q$ there is exactly one $(a_1, \ldots, a_{n-2})^{-1} \in Q$ such that for every $x \in Q$ the following equalities hold*

$$A(x, a_1^{n-2}, (a_1, \ldots, a_{n-2})^{-1}) = x \text{ and}$$
$$A((a_1, \ldots, a_{n-2})^{-1}, a_1^{n-2}, x) = x.[11]$$

In Proposition 6.2.1 as well as in Proposition 6.2.2, the n-semigroup (Q, A), $n \geq 3$, is followed by an $(n-2)$-ary operation in Q. In addition, if the fact that the mentioned $(n-2)$-ary operation in Proposition 6.2.1 generalizes the notion of a neutral element of a groupoid (:2.2.2) is omitted and in Proposition 6.2.2 its denotation [:$(\ldots)^{-1}$] is ignored, then we can say that they represent the same proposition. [The author expresses his thanks to K. Glazek for the information about the paper [5].]

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ON FORMULA DEPTH OF WEAKLY O-MINIMAL STRUCTURES

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1 Finite depth and strong monotonicity

Definition 1.1 (Dickmann)

- A weakly o-minimal structure (w.o.m. structure for short) is a totally ordered structure $M = (M, <, \dots)$ such that any definable subset of M is a finite union of convex disjoint sets under the ordering $<$.
- A theory is weakly o-minimal if all of its models are w.o.m.

Let M be a weakly o-minimal structure and let $\Phi(x,y)$ be a M-definable formula.

Then we write $\Phi(M,y_1) \geq \Phi(M,y_2)$, if $M \models \forall x_2 \exists x_1 [\bigwedge_{i=1}^{2} \Phi(x_i, y_i) \rightarrow x_2 \leq x_1]$

Then $=$ for $\Phi(M,y)$ is $(\leq) \wedge (\geq)$, and $<$ is $(\leq) \wedge (\neq)$.

We say that $\Phi(M,y)$ is strictly increasing on a set I, if $\forall y \forall z [y, z \in I \wedge y < z \rightarrow \Phi(M,y) < \Phi(M,z)]$.

If $E(y,z)$ is an equivalence relation on a set I, then $\Phi(M,y)$ is strictly increasing on a set $I/_E$, if $\forall y \forall z [y, z \in I \wedge \neg E(y,z) \wedge y < z \rightarrow \Phi(M,y) < \Phi(M,z)]$.

Strictly decreasing and constancy are defined like strictly increasing.

Let us assume that $\mathrm{dom}(\Phi(M,y)) = \{y \in M | M \models (\exists x)\Phi(x,y)\}$

Definition 1.2 [MMS] If M is a totally ordered structures, $\Phi(x,y)$ is a M-definable formula, $I \subset \mathrm{dom}(\Phi(M,y))$, then we say that Φ is tidy on I, if one of the following holds:

1. $\forall x \in I$ there is an infinite interval $J \subset I$ such, that $x \in J$ and Φ is strictly increasing on J (we say, that Φ is locally increasing on I).

2. $\forall x \in I$ there is an infinite interval $J \subset I$ such, that $x \in J$ and Φ is strictly decreasing on J (we say, that Φ is locally decreasing on I).

3. $\forall x \in I$ there is an infinite interval $J \subset I$ such, that $x \in J$ and Φ is constant on J (we say, that Φ is locally constant on I).

and, if for some $x \in I$ the set $\{y \in I \mid \Phi(M,t)$ is strictly monotonic on $(x,y) \cup (y,x)\}$ has maximum or minimum, then $\Phi(M,t)$ is strictly monotonic on I.

Definition 1.3 [1]If Φ and I like in definition 1.2, then we say that Φ is **n-tidy** on I, if the following holds:

- $\forall z \forall y \forall t\, [\Phi(M,z) = \Phi(M,y) \wedge z < t < y \rightarrow \Phi(M,z) = \Phi(M,t)]$
- $\Phi^{(n)}$ is tidy on I/E_{n-1}, where $\Phi^{(n)}(x,y) := \exists z[E_{n-1}(y,z) \wedge \Phi(x,z)]$
- $(\forall y \in I)\, E_n(I,y)/E_{n-1}$ has no minimum and maximum.
- $|I/E_n| \geq \omega$.

Where E_n is an equivalence relation on I such that

$$\begin{aligned} E_n(z,y) \;\Leftrightarrow\; & E_{n-1}(z,y) \vee \\ \vee\; & [[z < y \wedge \neg E_{n-1}(z,y) \rightarrow \Phi^{(n)} \upharpoonright [z,y]/E_{n-1} \text{ is strictly monotonic}] \wedge \\ \wedge\; & [y < z \wedge \neg E_{n-1}(z,y) \rightarrow \Phi^{(n)} \upharpoonright [y,z]/E_{n-1} \text{ is strictly monotonic}]] \end{aligned}$$

Here 0-tidy is tidy, $\Phi^{(0)} = \Phi$, $E_0(z,y) \Leftrightarrow z = y$.

Definition 1.4 If Φ and I like in definition 1.2, then we say that Φ is **strongly tidy** on I if there exists $n \in N$ such that Φ is $(n-1)$-tidy on I and $\Phi^{(n)}$ is strictly monotonic on I/E_{n-1}. So we say that the depth of formula Φ on I equals n.

Definition 1.5 Let M a be w.o.m. structure. We say that M has **strong monotonicity** (monotonicity [MMS]), if the following holds: whenever formula

[1]the idea of this notion was arising during of discussion with B. Baizhanov, whom the author thanks

$\Phi(x,y,a)$, $a \in M$, there is $m \in N$ and a partition of $\mathrm{dom}(\Phi(M,y,\bar{a}))$ into definable sets $X, I_1, \ldots, I_m$ such that X is finite, each I_i is convex and on each I_i formula $\Phi(x,y,\bar{a})$ is strongly tidy (tidy).

Definition 1.6 Let M be a w.o.m. structure. Then we say that M has **finite depth**, if the following holds: whenever formula $\Phi(x,y,\bar{z})$, there exists $n \in N$ such that for any $\bar{a} \in M$ and for any convex set $I \subset \mathrm{dom}(\Phi(M,y,\bar{a}))$, on which $\Phi(M,y,\bar{a})$ is strongly tidy, depth of $\Phi(M,y,\bar{a})$ on I is less than n.

Theorem 1.7 [MMS] *If all models of* $\mathrm{Th}(M)$ *are w.o.m., then M has monotonicity.*

Theorem 1.8 *If all models of* $\mathrm{Th}(M)$ *are w.o.m., then M has strong monotonicity and finite depth.*

Proof. Consider a formula $\Phi(M,y,\bar{z})$.

1. Assume that M has no finite depth, then for each $n \in N$ there exists $\bar{a} \in M$ and a definable convex set I_n such that $\Phi(M,y,\bar{a})$ is n-tidy on I_n.

Consider the formula $F(y,b,\bar{a}) := \Phi(M,y,\bar{a}) > \Phi(M,b,\bar{a})$. Clearly, that for any $b \in I_n$, $F(M,b,\bar{a})$ is a union at least $(n-1)$ $\neg F(M,b,\bar{a})$-separable convex sets. Thus there exists a $N \succ M$, there are $\bar{\alpha}, \beta \in N$ such that $F(M,\beta,\bar{\alpha})$ is a infinite union of disjoint convex sets. So, $\mathrm{Th}(M)$ is not weakly o-minimal.

2. Prove that M has strong monotonicity. Firstly, show that there exists a partition of $\mathrm{dom}(\Phi(M,y,a))$ into a finite union of definable convex sets such that on each set the following holds:

$$\forall z \forall y \forall t\, [\Phi(M,z,\bar{a}) = \Phi(M,y,\bar{a}) \wedge (z < t < y) \rightarrow [\Phi(M,t,\bar{a}) = \Phi(M,z,\bar{a})]] \tag{1}$$

Consider the following formulas:

$$\begin{aligned} \Theta_n(z) \;&:=\; \exists_y^{!n}[\bigwedge_{i=1}^{n}(y_i < z \wedge \Phi(M,y_i,\bar{a}) = \Phi(M,z,a) \wedge \neg E(y_i,z)) \wedge \\ &\quad \wedge \bigwedge_{i \neq j} \neg E(y_i,y_j)] \text{ где} \\ E(z,y) \;&:=\; (z,y \in \mathrm{dom}(\Phi)) \wedge \Phi(M,z,\bar{a}) = \Phi(M,y,\bar{a}) \wedge \\ &\quad \wedge \forall t[z \leq t \leq y \vee y \leq t \leq z \rightarrow \Phi(M,t,a) = \Phi(M,z,a)] \end{aligned}$$

Since $\mathrm{Th}(M)$ is weakly o-minimal, there is $n \in N$ such that $\Theta_{n+1}(M) = \emptyset$. Then $\Theta_0(M), \Theta_1(M), \ldots, \Theta_n(M)$ is the partition of $\mathrm{dom}(\Phi(M, y, \bar{a}))$, and for each $i \leq n$ on $\Theta_i(M)$ the property 1 holds.

Since $\Theta_i(M)$ is a finite union of convex sets, we can suppose, that it is convex. So we may assume that $\mathrm{dom}(\Phi(M, y, a))$ is convex and satisfies property 1.

By theorem 1.7 there is a $m < \omega$ and a partition of $\mathrm{dom}(\Phi(M, y, a))$ into definable sets $X, I_1, \ldots, I_n$, where X is finite, each I_i is convex and on each I_i formula $\Phi(M, y, a)$ is tidy. So, we can suppose that $\Phi(M, y, a)$ is tidy on $\mathrm{dom}(\Phi(M, y, \bar{a}))$.

Let $E_1(x, y)$ be an equivalence relation defined on $\mathrm{dom}(\Phi(M, y, a))$ as in definition 1.3. If $|\mathrm{dom}(\Phi(M, y, a))/E_1| < \omega$ then the theorem holds.

Assume that it is not true. It is possible to see, that the number of such elements $x \in \mathrm{dom}(\Phi(M, y, \bar{a}))$, that $E_1(M, x)$ has maximum or minimum is finite, consequently we can suppose that the condition 3. of the definition 1.3 for 1-tidy formula holds. Note, that the structure

$$(\mathrm{dom}(\Phi(M, y, a))/E_1, =, <, F^{(1)}(M, y, a))$$

where $F^{(1)}(z, y, \bar{a}) := \Phi^{(1)}(M, y, \bar{a}) > \Phi^{(1)}(M, z, a)$ and $\Phi^{(1)}(M, y, a)$ is defined as in the definition 1.3, has weakly o-minimal theory, then we may assume that $\Phi^{(1)}$ is tidy on $\mathrm{dom}(\Phi)/E_1$.

Since M has a finite depth there is $n \in N$ such that at most $\Phi^{(n)}(M, y, a)$ is strictly monotonic on $\mathrm{dom}(\Phi(M, y, a))/E_n$. So, it is clear that the theorem holds.

2 Examples of w.o.m. structures

We illustrate in this part introduced notion of depth by two examples of w.o.m. structures without w.o.m. theory. Depth of any formula of first structure is zero. Second structure has no finite depth but it has monotonicity.

First example. Let $M = N \times Q$, $\sigma = \{=, <, R^2\}$;

Order is lexicographical, relation R is defined as follows:

$$R(M, M) = \bigcup_{n<\omega} K_n,$$

$$\text{where } K_n = \{((k, q), (k, p)) \mid k \geq n,\ p = 2^n q \text{ or } q = 2^n p\}$$

Clearly, that $\mathrm{Th}(M)$ is not w.o.m.

We say that (C, D) is a cut of M if $C \cup D = M$, $C \cap D = \emptyset$ and $\forall c \in C\ \forall d \in D\ (c < d)$.

Theorem 2.1 [K] *Let M be a totally ordered structure. Then M is weakly o-minimal iff the following holds on M:*

1. *Any cut (C, D) in M has at most two complete 1-types over M extending (C, D).*
2. *If a cut (C, D) in M has two complete 1-types over M extending (C, D), then the set of realizations of each of these types is convex in any elementary extension of M.*

Theorem 2.2 *M is weakly o-minimal.*

Proof. Define firstly the following formulae:

$$\begin{aligned} L_0(x) &:= \exists! y\, [R(x, y)] \ldots \\ L_n(x) &:= \exists^{!(2n+1)}_y\, [R(x, y)] \ldots \end{aligned}$$

$$\psi(x) = y \iff M \models R(x, y) \wedge (x < y) \wedge \forall z[(x < z) \wedge R(x, z) \to (z \geq y)]$$

To prove Theorem 2.2 we apply Theorem 2.1. Let (C, D) be a cut in M, A be a finite subset of M.

Case 1. $C \neq M$ and $C \neq \bigcup_{i \leq n} L_i(M)$ for each $n < \omega$.

If $\{a \in \mathrm{dcl}(A) \mid a < b\} = \{a \in \mathrm{dcl}(A) \mid a < c\}$ then there exists $f \in \mathrm{Aut}(M/A)$ such that $f(b) = c$, so the cut (C, D) in M has only one complete 1 types over M extending (C, D). So, Theorem for this case obviously holds.

Case 2. $C = \bigcup_{i \leq p} L_i(M)$ for some $p < \omega$.

Let $0\text{-dcl}(A) = A$, $n\text{-dcl}(A) = \{x \in M \mid \exists y \in (n-1)\text{-dcl}(A) : M \models R(x, y)\}$.

It is clear, that for any $A \subset M$ if $A \cap L_p(M) = \emptyset$ then for any $b, c \in L_p(M)$ $tp(c/A) = tp(b/A)$. So, we can suppose that $A \subset L_p(M)$.

For $m \leq \omega$ a game $G_m(A, B)$ played by two players, duplicator and spoiler, consists of m turns. On move k, Spoiler chooses an element from A or B (denote by a_k or b_k respectively). Duplicator then chooses an element from the other structures (denote by b_k or a_k respectively). Duplicator wins if the sequences $\langle a_i : i < m\rangle$ and $\langle b_i : i < m\rangle$ are partially isomorphic. So, we say that $A \simeq_m B$ if duplicator has a winning strategy for the family of games: $G_m(A, B)$.

Lemma 2.3 *Let $b, c \in L_p(M)$, let $A \subset L_p(M)$ be finite and for all $a \in (2^{n+1})$-$\mathrm{dcl}(A)$ we have that $b > a$ and $c > a$, then $(L_p(M), c, a)_{a\in A} \simeq_n (L_p(M), b, a)_{a\in A}$.*

Proof of Lemma. Let $C = (L_p(M), c, a)_{a\in A}$ and $B = (L_p(M), b, a)_{a\in A}$, $C_0 = \{c\} \cup A$, $B_0 = \{b\} \cup A$, $C_{i+1} = \{c_{i+1}\} \cup C_i$, $B_{i+1} = \{b_{i+1}\} \cup B_i$.

Suppose that $C_i \cong B_i$ by partial isomorphism g_i. We can assume that after the move i $\forall k, j \leq |C_i|\ [k < j \to (c_k < c_j \wedge b_k < b_j)]$.

Without loss of generality we can assume that Spoiler chooses $c_{i+1} \in C$ and $c_j < c_{i+1} < c_{j+1}$ for some $0 < j \leq |C_i|$ (here c_0 is $-\infty$, $c_{|C_i|+1}$ is $+\infty$).

Define a function $f(c_{i+1}) = k$ if there are $x, y \in k$-$\mathrm{dcl}(c)$ for some $c \in C_i$ such that $x \leq c_{i+1} \leq y$ and for any $c' \in C_i$, for any $t, z \in (k-1)$-$\mathrm{dcl}(c)$ it is not hold that $t \leq c_{i+1} \leq z$ We say that $h(c_{i+1}) = c$ if c is such element. Obviously, that this c may be c_j or c_{j+1}. We can suppose that $c = c_j$ and if $\phi^l(c_j) \leq c_{i+1} \leq \phi^{l+1}(c_j)$ and $\phi^{-m-1}(c_{j+1}) \leq c_{i+1} \leq \phi^{-m}(c_{j+1})$ then $l \leq m$.

Define also functions: let x, y are as above and for any $z \in k$-$\mathrm{dcl}(A)$ we have $z \notin (x, y)$, so we say that $v_1(c) = x$ and $v_2(c) = y$, where $c = h(c_{i+1})$.

Let Duplicator take b_{i+1} which must satisfy the following conditions:

1. $c_{i+1} \in k$-$\mathrm{dcl}(h(c_{i+1}))$ and $k \leq 2^{n-i+1} \Rightarrow b_{i+1} = h^{-1}(g_i(h(c_{i+1})))$;
2. $f(c_{i+1}) \leq 2^{n-i+1} \Rightarrow v_1(g_i(h(c_{i+1}))) < b_{i+1} < v_2(g_i(h(c_{i+1})))$;
3. $f(c_{i+1}) > 2^{n-i+1} \Rightarrow \exists b'$ such that $t < b' < z$ for any $t \in 2^{n-i+1}$-$\mathrm{dcl}(g_i(c_j))$ and for any $z \in 2^{n-i+1}$-$\mathrm{dcl}(g_i(c_{j+1}))$. And so, we take some such element as b_{i+1}.

Clearly that $b_{i+1} > b_j$ and if $C_{i+1} \models R(c_{i+1}, c_s)$ then $B_{i+1} \models R(b_{i+1}, b_s)$. Show that $b_{i+1} < b_{j+1}$. Suppose that $k = f(c_{i+1})$.

1. $k \leq 2^{n-i+1} \Rightarrow b_{j+1} \in k$-$\mathrm{dcl}(b_j) \setminus (k-1)$-$\mathrm{dcl}(b_j)$. So, it is easy to see, that $b_{i+1} < b_{j+1}$.
2. $k > 2^{n-i+1} \Rightarrow b_{j+1} \notin 2^{n-i+1}$-$\mathrm{dcl}(b_j)$. Thus, it is clear that $b_{i+1} < b_{j+1}$.

$\Rightarrow C_{i+1} \cong B_{i+1}$. □

We have that $\{c < x < d \mid c \in C,\ d \in D\} \cup \{L_p(x)\}$ has only one complete 1-types over M extending it. $\{c < x < d \mid c \in C,\ d \in D\} \cup \{\neg L_p(x)\}$ has only one complete 1-types over M extending it too (the proof is analogous). The set of realizations of each of these types is convex in any elementary extension of M. So, Theorem for this case holds.

Case 3. $C = M$.

It follows from the cases 1 and 2 that $\forall A \subset M$, if $\exists x \forall a \in A[L_n(x) \wedge a < x]$, then for any b, c such that $M \models \forall x(L_n(x) \to x < c \wedge x < b)$ the following holds: $tp(b/A) = tp(c/A) \iff tp(b/\emptyset) = tp(c/\emptyset)$.

To prove Theorem for this case it is sufficient to show the following lemma.

Lemma 2.4 *Let $b, c \in M$, $M \models \forall x[L_{2^{n+1}}(x) \to x < b \wedge x < c]$, then $(M, c) \simeq_n (M, b)$.*

Proof. For simplification, we partition this game into two parts, and then it is easy to prove the lemma by combining of them.

Remark 2.5 *Let $b, c \in N$ and $c, b > 2^n$. Then $(N, <, c) \simeq_n (N, <, b)$.*

Proof. Let $C = (N, <, c)$ and $B = (N, <, b)$, $C_0 = \{c\}$, $B_0 = \{b\}$, $C_{i+1} = \{c_{i+1}\} \cup C_i$, $B_{i+1} = \{b_{i+1}\} \cup B_i$.

We can assume that after the move i $\forall k, j \leq i[k < j \to c_k < c_j \wedge b_k < b_j]$. Define a function $h(b_{i+1}) = j$, where $-1 \leq j \leq i$ if $|b_{i+1} - b_j| \leq |b_{i+1} - b_k|$, $-1 \leq k \leq i$, where $b_0 = b$, $b_{-1} = 0$ and j is minimal. Define $h(c_{i+1})$ like $h(b_{i+1})$.

Suppose that $C_i \cong B_i$. Without loss of generality we can assume that Spoiler chooses $c_{i+1} \in C$, moreover we may assume that $c_{i+1} < c_j$, where $j = h(c_{i+1})$ (other cases are proved analogiuosly).

1. $|c_{i+1} - c_j| \le 2^{n-i} \Rightarrow b_{i+1} = b_j + c_{i+1} - c_j$;
2. $|c_{i+1} - c_j| > 2^{n-i} \Rightarrow \exists b_{i+1} \in (b_{j-1} + 2^{n-i}, b_j - 2^{n-i})$, and we take some such element as b_{i+1}.

Clearly that $b_{i+1} < b_j$. Show that $b_{i+1} > b_{j-1}$.

1. $|c_{i+1} - c_j| \le 2^{n-i}$
   (a) $c_j - c_{j-1} \le 2^{n-i+1} \Rightarrow b_j - b_{j-1} = c_j - c_{j-1} \Rightarrow b_{j-1} < b_{i+1}$;
   (b) $c_j - c_{j-1} > 2^{n-i+1} \Rightarrow b_j - b_{j-1} \ge 2^{n-i+1} \Rightarrow b_{j-1} < b_{i+1}$;
2. $|c_{i+1} - c_j| > 2^{n-i} \Rightarrow b_j - b_{j-1} > 2^{n-i+1} \Rightarrow b_{j-1} < b_{i+1}$.

$\Rightarrow C_{i+1} \cong B_{i+1}$. □

Remark 2.6 *Let $C = \langle L_l(M), <, R\rangle$, $B = \langle L_m(M), <, R\rangle$, where $m, l > 2^{n+1}$. Then $C \simeq_n B$.*

Proof. Move 1. Suppose that Spoiler takes some element $c_1 \in C$, then Duplicator takes arbitrary $b_1 \in B$. Clearly that $B_1 \cong C_1$, where $C_1 = \{c_1\}$, $B_1 = \{b_1\}$.

Move (i+1). Suppose that $C_i \cong B_i$ by some partial isomorphism g_i. We may assume that $\forall k, j \le i[k < j \Rightarrow b_k < b_j \wedge c_k < c_j]$. Without loss of generality we can assume that Spoiler chooses $c_{i+1} \in C$ and Duplicator — b_{i+1}.

Define the following functions $f(x, b) = p$ and $h(x, b) = s$, where p and s are such that

$$M \models \exists^{!p}_{y}[\bigwedge_{i=1}^{p} R(x, y_i) \wedge (x < y_i < b \vee b < y_i < x)]$$

$$M \models \exists^{!s}_{y}[\bigwedge_{i=1}^{s} R(x, y_i) \wedge (x < b < y_i \vee y_i < b < x)]$$

Assume that $c_{i+1} \in (c_j, c_{j+1})$, and if $\phi^k(c_j) < c_{i+1} \le \phi^{k+1}(c_j)$ and $\phi^{-K-1}(c_{j+1}) \le c_{i+1} < \phi^{-K}(c_{j+1})$ then $k \le K$ (the contrary case is proved analoginosly).

Let $c_j^{+,N} = \max\{x \mid x \in N\text{-dcl}(c_j)\}$, $c_j^{-,N} = \min\{x \mid x \in N)\text{-dcl}(c_j)\}$;

So, Duplicator must take $b_{i+1} > b_j$ which satisfies the following conditions:

1. $c_{i+1} \in \text{dcl}(c_j) \Rightarrow b_{i+1} \in \text{dcl}(b_j)$;
1.1. $c_{i+1} \in 1\text{-dcl}(c_j) \Rightarrow b_{i+1} \in 1\text{-dcl}(b_j)$;
1.1.1. $f(c_j, c_{i+1}) \le 2^{n-i+1} \Rightarrow f(c_j, c_{i+1}) = f(b_j, b_{i+1})$;
1.1.2. $h(c_j, c_{i+1}) \le 2^{n-i+1} \Rightarrow h(c_j, c_{i+1}) = h(b_j, b_{i+1})$;
1.1.3. $f(c_j, c_{i+1}) > 2^{n-i+1}$ and $h(c_j, c_{i+1}) > 2^{n-i+1} \Rightarrow f(b_j, b_{i+1}) > 2^{n-i+1}$ and $h(b_j, b_{i+1}) > 2^{n-i+1}$. If for any $q \le n - i + 1$ and for any $c_v^{*,q} \in q\text{-dcl}(C_i)$ we have that $C \models R(c_v^{*,q}, c_{i+1})$ where $* \in \{+, -\}$ then
1.1.3.1. $f(c_v^{*,q}, c_{i+1}) \le 2^{n-i-q+1}$ then $f(b_v^{*,q}, b_{i+1}) = f(c_v^{*,q}, c_{i+1})$;
1.1.3.2. $h(c_v^{*,q}, c_{i+1}) \le 2^{n-i-q+1}$ then $h(b_v^{*,q}, b_{i+1}) = h(c_v^{*,q}, c_{i+1})$
1.1.3.3. $f(c_v^{*,q}, c_{i+1}) > 2^{n-i-q+1}$ and $h(c_v^{*,q}, c_{i+1}) > 2^{n-i-q+1} \Rightarrow f(b_v, b_{i+1}) > 2^{n-i-q+1}$ and $h(b_j, b_{i+1}) > 2^{n-i-q+1}$.
1.1.3.4. If for any $q \le n - i + 1$ and for any $c_v^{*,q} \in q\text{-dcl}(C_i)$ we have that $C \models R(c_v^{*,q}, x) \wedge x < c_{i+1} < \phi(x)$ where $* \in \{+, -\}$ then we find $x' \in B$ using 1.1.3.1., 1.1.3.2., 1.1.3.3. and think that $x' < b_{i+1} < \phi(x')$.

1.N. $c_{i+1} \in N\text{-dcl}(c_j) \setminus (N-1)\text{-dcl}(c_j) \Rightarrow b_{i+1} \in N\text{-dcl}(b_j) \setminus (N-1)\text{-dcl}(b_j)$;
1.N.1. $f(c_j^{+,N-1}, c_{i+1}) \le 2^{n-i-N+2} \Rightarrow f(c_j^{+,N-1}, c_{i+1}) = f(b_j^{+,N-1}, b_{i+1})$;
1.N.2. $h(c_j^{+,N-1}, c_{i+1}) \le 2^{n-i-N+2} \Rightarrow h(c_j^{+,N-1}, c_{i+1}) = h(b_j^{+,N-1}, b_{i+1})$;
1.N.3. $f(c_j^{+,N-1}, c_{i+1}) > 2^{n-i-N+2}$ and $h(c_j^{+,N-1}, c_{i+1}) > 2^{n-i-N+2} \Rightarrow f(b_j^{+,N-1}, b_{i+1}) > 2^{n-i-N+2}$ and $h(b_j^{+,N-1}, b_{i+1}) > 2^{n-i-N+2}$. And if for any $q \le n - i - N + 2$ and for any $c_v^{*,q} \in q\text{-dcl}(C_i)$ we have that $C \models R(c_v^{*,q}, c_{i+1})$ where $* \in \{+, -\}$ then
1.N.3.1. $f(c_v^{*,q}, c_{i+1}) \le 2^{n-i-N-q+2}$ then $f(b_v^{*,q}, b_{i+1}) = f(c_v^{*,q}, c_{i+1})$;
1.N.3.2. $h(c_v^{*,q}, c_{i+1}) \le 2^{n-i-N-q+2}$ then $h(b_v^{*,q}, b_{i+1}) = h(c_v^{*,q}, c_{i+1})$
1.N.3.3. $f(c_v^{*,q}, c_{i+1}) > 2^{n-i-N-q+2}$ and $h(c_v^{*,q}, c_{i+1}) > 2^{n-i-N-q+2} \Rightarrow f(b_v^{*,q}, b_{i+1}) > 2^{n-i-q-N+2}$ and $h(b_j^{*,q}, b_{i+1}) > 2^{n-i-q-N+2}$.
1.N.3.4. If for any $q \le n - i - N + 2$ and for any $c_v^{*,q} \in q\text{-dcl}(C_i)$ we have that $C \models R(c_v^{*,q}, x) \wedge x < c_{i+1} < \phi(x)$ where $* \in \{+, -\}$ then we find $x' \in B$ using 1.N.3.1., 1.N.3.2., 1.N.3.3. and think that $x' < b_{i+1} < \phi(x')$.

2. $\exists k < \omega$ such that $\phi^k(c_j) < c_{i+1} < \phi^{k+1}(c_j)$ then we consider $\phi^k(c_j)$ as c_{i+1} and we find some b for $\phi^k(c_j)$ using 1.. So $b < b_{i+1} < \phi(b)$. And if for $c_z \in C_i$, $c_{i+1} \in n\text{-dcl}(c_z)$, where $n \le n - i + 1$ then to define more precisely b_{i+1} we repeat 1. using c_z as c_j.

Let $C_{i+1} = \{c_{i+1}\} \cup C_i$, $B_{i+1} = \{b_{i+1}\} \cup B_i$.

Prove that $C_{i+1} \cong B_{i+1}$. We need to prove that $b_{i+1} < b_{j+1}$ and $C_{i+1} \models R(c_w, c_{i+1})$ iff $B_{i+1} \models R(b_w, c_{i+1})$ for each $w \leq i$.

First, show that $b_{i+1} < b_{j+1}$. Let $c_{i+1} \in [x, \phi(x)]$ where $x \in k_1\text{-dcl}(c_j) \setminus (k_1 - 1)\text{-dcl}(c_j)$ and $c_{i+1} \in [y, \phi(y)]$ where $y \in k_2\text{-dcl}(c_{j+1}) \setminus (k_2 - 1)\text{-dcl}(c_{j+1})$, consequently $c_{j+1} \in [z, \phi(z)]$ where $z \in (k_1 + k_2 - 1)\text{-dcl}(c_j) \setminus (k_1 + k_2 - 2)\text{-dcl}(c_j)$. We have by our strategy that $b_{i+1} \in [x', \phi(x')]$ with $x' \in k_1\text{-dcl}(b_j) \setminus (k_1 - 1)\text{-dcl}(b_j)$ and $b_{j+1} \in [z', \phi(z')]$ with $z' \in (k_1 + k_2 - 1)\text{-dcl}(b_j) \setminus (k_1 + k_2 - 2)\text{-dcl}(b_j)$. It means that $b_{i+1} < b_{j+1}$

Show, that $C_{i+1} \models R(c_q, c_{i+1}) \Rightarrow B_{i+1} \models R(b_q, b_{i+1})$ (the reverse statement is proved analoginosly).

Let $C_{i+1} \models x \leq c_{i+1} < \phi(x)$, where $x = \phi^k(c_j)$. To simplify the proof we can suppose that $x = c_{i+1}$.
a. $c_q < c_j$, then $C_{i+1} \models R(c_q, c_j)$;
a.1. $f(c_q, c_j) \leq 2^{n-i+2} \Rightarrow f(b_q, b_j) = f(c_q, c_j)$.
a.1.1. $f(c_j, c_{i+1}) \leq 2^{n-i+1}$. Then $f(b_q, b_{i+1}) = f(c_q, c_{i+1})$. Thus $B_{i+1} \models R(b_q, b_{i+1})$;
a.1.2. $h(c_j, c_{i+1}) \leq 2^{n-i+1}$. Then $h(b_q, b_{i+1}) = f(b_q, b_j) + h(b_j, b_{i+1}) + 1 = f(c_q, c_j) + h(c_j, c_{i+1}) + 1 = h(c_q, c_{i+1})$. So $B_{i+1} \models R(b_q, b_{i+1})$;
a.1.3. It follows from a.1.1. and a.1.2.

a.2. $h(c_q, c_j) \leq 2^{n-i+2} \Rightarrow h(b_q, b_j) = h(c_q, c_j)$. Then $h(c_j, c_{i+1}) \geq 2^l - 2^{n-i+2} \geq 2^{n+1} - 2^{n-i+2} > 2^n$, because $i \geq 3$ (l is from the conditions of this remark).
a.2.1. $f(c_j, c_{i+1}) \leq 2^{n-i+1}$. Then $f(b_j, b_{i+1}) = f(c_q, c_{i+1})$ and $h(b_q, b_{i+1}) = h(b_q, b_j) - f(b_j, b_{i+1}) - 1 = h(c_q, c_j) - f(c_j, c_{i+1}) - 1 = h(c_q, c_{i+1})$. So, $B_{i+1} \models R(b_q, b_{i+1})$;
a.2.2. It is impossible.
a.2.3. It follows from a.1.1. and a.1.2.

a.3. $h(c_q, c_j) > 2^{n-i+2}, f(c_q, c_j) > 2^{n-i+2}$. It easy follows from a.1. and a.2..

b. $c_q > c_{i+1} > c_j \Rightarrow b_q > b_{i+1} > b_j$. If $M \models R(c_j, c_q) \Rightarrow M \models R(b_j, b_q) \Rightarrow M \models R(b_q, b_{i+1})$. So, assume contrary, then $c_j \in 2\text{-dcl}(c_q)$, consequently $b_j \in 2\text{-dcl}(c_q)$.
b.1. $f(c_j, c_{i+1}) \leq 2^{n-i+1}$ then $f(c_j^{+,1}, c_q) \leq f(c_j, c_{i+1})$, thus $f(b_j^{+,1}, b_q) \leq f(b_j, b_{i+1})$. So $M \models R(b_q, b_{i+1})$.
b.2. $h(c_j, c_{i+1}) \leq 2^{n-i+1}$ then $f(c_q, c_{i+1}) > h(c_j, c_{i+1})$.
b.2.1. $f(c_q, c_{i+1}) \leq 2^{n-i+1}$ then $f(c_j^{+,1}, c_q) = f(c_q, c_{i+1}) - f(c_j^{+,1}, c_{i+1}) \leq 2^{n-i+1}$. So, $f(b_j^{+,1}, b_q) = f(c_j^{+,1}, c_q)$. Thus it is clear that $B_{i+1} \models R(b_{i+1}, b_q)$;
b.2.2. $h(c_q, c_{i+1}) \leq 2^{n-i+1}$ If $M \models \neg R(b_q, b_{i+1})$ then $h(b_q, b_j^{+,1}) < f(b_j^{+,1}, b_{i+1}) = h(b_j, b_{i+1}) + 1 = h(c_j, c_{i+1}) + 1$, that is $h(b_q, b_j^{+,1}) \leq 2^{n-i+1}$ consequently $h(b_q, b_j^{+,1}) = h(c_q, c_j^{+,1}) < h(c_j, c_{i+1})$, thus $C_{i+1} \models \neg R(c_q, c_{i+1})$. It is a cotradiction.
b.2.3. it follows from b.2.1. and b.2.2..

b.3. $h(c_j, c_{i+1}) > 2^{n-i+1}$ and $f(c_j, c_{i+1}) > 2^{n-i+1}$. It follows from 1.1.3.1. or 1.1.3.2. or 1.1.3.3..

□

Describe the winning strategy for the proof of Lemma 2.4. Let Spoiler take some element $c_{i+1} \in (M, c)$, in particulary $c_{i+1} \in L_n(M)$, $n < \omega$. If $L_n(M) \cap C_i = \emptyset$ then Duplicator takes arbitrary $b_{i+1} \in L_k(M)$, where k is defined by Remark 2.5. If some $c' \in L_n(M)$ then Duplicator takes $b_{i+1} \in L_k(M)$, where k is such that $g_i(c') \in L_k(M)$ and b_{i+1} must satisfy the demands described in Remark 2.6.

□

So, when $C = M$ we have that the cut (C, D) in M has only one complete 1 types over M extending (C, D). So, Theorem for this case holds. Thus Theorem is proved.

Lemma 2.7 *There is no M-defaneble equivalence relation on M with infinite number of infinite classes.*

Proof. Assume contrary. Let $E'(x, y)$ be a M-definable equivalence relation on $I' \subseteq M$ with infinite number of infinite classes.

Let $E(x, y) \iff E'(x, y) \wedge \forall t[x < t < y \vee y < t < x \to E(x, t)]$;
$I = \{x \in I' \mid E(M, x) \neq \{x\}\}$.

So, any class of $E(x, y)$ is convex and infinite and number of classes is infinite. Since M is w.o.m. it is obviously that the number of classes such that they have bound in M is finite. Thus we may suppose that any class of $E(x, y)$ has no bound in M.

Assume that for some $n < \omega$ there exists $a, b \in L_n(M) \cap I$ such that

$M \models \neg E(a,b)$. Let $C = \{x \in M \mid \exists y \in E(M,a)(x < y)\}$, $D = M \setminus C$. So $\{c < x < d \mid c \in C, d \in D\} \cup \{E(x,a)\}$ and $\{c < x < d \mid c \in C, d \in D\} \cup \{\neg E(x,a)\}$ are consistent, but from theorem 2.2 case 1. it follows that (C,D) has the only complete 1-types over M extending (C,D). It is a cotradiction. Consequently for each $n < \omega$ for each $a \in L_n(M)$ we have $L_n(M) \subseteq E(M,a)$. It is easy to see by Lemma 2.4 and Lemma 2.3 that there exists $n < \omega$ such that $L_n(M) \cap I \neq \emptyset$ and for any $a \in L_n(M)$ $(a,+\infty) \subset E(a,M)$. Hence $E(x,y)$ has only finite number of classes, but it is a contradiction. □

Corollary 2.8 *Depth of any M-definable formula $\Phi(x,y)$ equals 0.*

Second example. Here we adduce the example of w.o.m structure without w.o.m. theory which has no finite depth but it has monotonicity.

$$\text{Let } M = \{-1,1\} \times \sum_{n<\omega} \{2n+1\} \times Q^{2n+1}, \quad \sigma = \{=,<,f^1\}$$

Order is lexicographical, unary function f is defined as follows:

$$f(s,2n+1,q_{2n+1},\dots,q_1) = (-s,2n+1,q_{2n+1},-q_{2n},\dots,q_3,-q_2,q_1)$$

It is clear, that M has monotonicity, because f is tidy on M. We prove that M is w.o.m., but $\mathrm{Th}(M)$ is not w.o.m.. To study M better we define the following formulae:

$$\begin{aligned}
E_{2n}(x,y) &:= E_{2n-1}(x,y) \vee \forall z_1 \forall z_2[(x < z_1 < z_2 < y \vee \\
&\vee\ \ y < z_1 < z_2 < x) \wedge \neg E_{2n-1}(z_1,z_2) \to f(z_1) > f(z_2)] \\
E_{2n+1}(x,y) &:= E_{2n}(x,y) \vee \forall z_1 \forall z_2[(x < z_1 < z_2 < y \vee \\
&\vee\ \ y < z_1 < z_2 < x) \wedge \neg E_{2n}(z_1,z_2) \to f(z_1) < f(z_2)] \\
\text{где} \quad & \quad E_0(x,y) \Leftrightarrow x = y \\
L_0(x) &:= f(x) > x \\
L_1(x) &:= \forall y[y < x \to E_1(x,y)] \quad \dots \\
L_{2n+1}(x) &:= \forall y[\forall z[L_{2n-1}(z) \to z < x \wedge z < y] \to \\
&\to\ \ (y < x \to E_{2n+1}(x,y))]
\end{aligned}$$

$$\begin{aligned}
R_0(x) &:= f(x) < x \\
R_1(x) &:= \forall y[y < x \wedge R_0(y) \to E_1(x,y)] \quad \dots \\
R_{2n+1}(x) &:= \forall y[\forall z[R_{2n-1}(z) \to z < x \wedge z < y] \to \\
&\to\ \ (y < x \to E_{2n+1}(x,y))] \quad \dots
\end{aligned}$$

Lemma 2.9 *M has no finite depth.*

Proof. It is easy to see, that on the sets $\bigcup\limits_{i=2n+1}^{\omega} L_i$, $\bigcup\limits_{i=2n+1}^{\omega} R_i$ function f is $(2n+1)$-tidy, this implies lemma.

Corollary 2.10 *$\mathrm{Th}(M)$ is not weakly o-minimal.*

Theorem 2.11 *M is weakly o-minimal.*

Proof. To prove Theorem 2.11 we apply Theorem 2.1.
Let (C,D) be a cut in M.
Case 1. $C \neq L_0(M)$, $C \neq M$. First, prove the following lemma.

Lemma 2.12 *Let $A \subset M$. Let $b, c \in M \setminus \mathrm{dcl}(A)$ are such that*

1. $\{a \in \mathrm{dcl}(A) \mid a < b\} = \{a \in \mathrm{dcl}(A) \mid a < c\}$;

2. $(\exists n > 0)\ M \models [L_n(c) \wedge L_n(b)] \vee [R_n(c) \wedge R_n(b)]$;

3. $(\exists k \in N)(\forall a \in A)\ M \models \neg E_{k-1}(a,b) \wedge \neg E_{k-1}(a,c)$ and $M \models E_k(b,c)$

then $tp(b/A) = tp(c/A)$.

Proof. Clearly, that $A \cup \{c\}$ and $A \cup \{b\}$ are partially isomorphic and this partial isomorphism is extended to automorphism of M. □

Let X be a convex set. Then $X^+ = \{y \in M \mid \forall x \in X(x < y)\}$.
So, if $C^+ \neq L_n(M)^+$, $C^+ \neq R_n(M)^+$ for any $n < \omega$ then the cut (C,D) in M has only one complete 1-types over M extending (C,D). If $C^+ = L_n(M)^+$ or $C^+ = R_n(M)^+$ for some $n > 0$, then $\{c < x < d \mid c \in C,\ d \in D\} \cup \{L_n(x)\}$ and $\{c < x < d \mid c \in C,\ d \in D\} \cup \{\neg L_n(x)\}$ have only one complete 1-types over M extending them, and the set of realizations of each of these types is convex in any elementary extension of M. So, Theorem for this case holds.

It follows from the lemma that $\forall A \subset M$, if $\exists x \forall a \in A[L_n(x) \wedge a < x]$, then for any b, c: $M \models L_0(b) \wedge L_0(c)$ and $\forall x(L_n(x) \rightarrow x < c \wedge x < b)$ the following holds: $tp(b/A) = tp(c/A) \Leftrightarrow tp(b/\emptyset) = tp(c/\emptyset)$. If we substitute R instead of L, then we take the same.

Case 2. $C = L_0(M)$ or $C = M$.

To prove Theorem for this case it is sufficient to show the following lemma.

Lemma 2.13 *Let $b, c \in L_0(M)$, $M \models \forall x[L_{2^{n+2}}(x) \rightarrow x < b \wedge x < c]$. Then $(M, c) \simeq_n (M, b)$.*

Proof. For simplification, we partition this game into two parts, and then it is easy to prove the lemma by combining of them. First part is a remark 2.5, second one is the following remark.

Remark 2.14 *Let $C = \langle L_l(M) \cup R_l(M), <, f\rangle$, $B = \langle L_m(M) \cup R_m(M), <, f\rangle$, where $m, l > 2^{n+1}$. Then $C \simeq_n B$.*

Proof. Move 1. Suppose that Spoiler takes some element $c_1 \in C$, then Duplicator takes $b_1 \in B$ such that $M \models L_l(c_1) \leftrightarrow L_m(b_1)$. Clearly that $B_1 \cong C_1$, where $C_1 = \{c_1, f(c_1)\}$, $B_1 = \{b_1, f(b_1)\}$.

Move (i+1). Suppose that $C_i \cong B_i$ by some partial isomorphism g_i. We may assume that $\forall k, j \leq 2i[k < j \Rightarrow b_k < b_j \wedge c_k < c_j]$. Without loss of generality we can assume that Spoiler chooses $c_{2i+1} \in C$ and Duplicator — b_{2i+1}.

Define the following functions $h(b_{2i+1}) = j$ and $h'(b_{2i+1}) = k$, where k and j is such that $M \models E_k(b_{2i+1}, b_j)$ and $(\forall b \in B_i)$ $M \models \neg E_{k-1}(b_{2i+1}, b)$ and there is no $b \in B_i$ lying between b_j and b_{2i+1}. $h(c_{2i+1}) = j$ and $h'(c_{2i+1}) = k$ are defined analogiuosly.

If $c_{2i+1} \in (c_s, c_{s+1})$, then $b_{2i+1} \in (b_s, b_{s+1})$ and $h(b_{2i+1}) = h(c_{2i+1})$.

1. If $h'(c_{2i+1}) \leq 2^{n-i+1}$, then $h'(b_{2i+1}) = h'(c_{2i+1})$
2. If $h'(c_{2i+1}) \geq l - 2^{n-i+1}$, then $m - h'(b_{2i+1}) = l - h'(c_{2i+1})$
3. If $2^{n-i+1} < h'(c_{2i+1}) < l - 2^{n-i+1}$, then $2^{n-i+1} < h'(b_{2i+1}) < m - 2^{n-i+1}$ and $h'(c_{2i+1}) + h'(b_{2i+1})$ is divisable on 2.

Let $C_{i+1} = C_i \cup \{c_{2i+1}, f(c_{2i+1})\}$ and $B_{i+1} = B_i \cup \{b_{2i+1}, f(b_{2i+1})\}$. Clearly, that $C_{i+1} \cong B_{i+1}$. □

Describe the winning strategy for proof of Lemma 2.13. Let Spoiler take some element $c_{i+1} \in (M, c)$, in particulary $c_{i+1} \in L_n(M)$, $n > 0$. If $L_n(M) \cap C_i = \emptyset$ then Duplicator takes arbitrary $b_{i+1} \in L_k(M)$, where k is defined by Remark 2.5. If some $c' \in L_n(M)$ then Duplicator takes $b_{i+1} in L_k(M)$, where $g_i(c') \in L_k(M)$ and b_{i+1} must satisfy the demands described in Remark 2.14. □

If $C = L_0(M)$ then $\{c < x < d \mid c \in C,\ d \in D\} \cup \{L_0(x)\}$ and $\{c < x < d \mid c \in C,\ d \in D\} \cup \{\neg L_0(x)\}$ have only one complete 1-types over M extending them, and the set of realizations of each of these types is convex in any elementary extension of M. If $C = M$ then the cut (C, D) in M has only one complete 1-types over M extending (C, D).

So, Theorem for this case holds. Thus Theorem is proved.

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ENGEL ELEMENTS IN GROUPS WITH THE MINIMAL CONDITION ON CENTRALIZERS

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Introduction

Nilpotency properties in groups with the chain condition on centralizers ($\mathcal{M}_c$) have been studied by a number of people, generalizing corresponding results for linear groups. While Bryant and Hartley dealt with the periodic case and proved that periodic locally nilpotent $\mathcal{M}_c$-groups are nilpotent-by-finite [2], and periodic locally soluble $\mathcal{M}_c$-groups are nilpotent-by-abelian-by-finite [3], non-periodic groups were first dealt with in an intermediate case, the class of *substable* groups, which lies in between the linear and the $\mathcal{M}_c$-groups. Local nilpotency properties of substable groups are well understood ([6, 7, 8, 4], see also [9]):

1. Uniformly locally nilpotent substable groups are nilpotent.
2. Locally nilpotent substable groups are hypercentral.
3. The bounded left Engel elements of a substable group generate the Fitting subgroup.
4. The unbounded left Engel elements of a substable group generate the Hirsch-Plotkin radical.

The question was asked to what extent substability can be replaced by $\mathcal{M}_c$ in the above results. Derakhshan could show that the Fitting subgroup of any $\mathcal{M}_c$-group is nilpotent; this quickly led to a generalization of 1. to the $\mathcal{M}_c$ case [4]. Recently, Bludov in a short, elegant paper [1], dealt with the $\mathcal{M}_c$-analogue of 2; he also found an independent proof of Derakhshan's result. In this paper we shall generalize 3.

1 Notation and Preliminaries

Our commutators are left-normalized, defined inductively via $[g,h] = g^{-1}h^{-1}gh$ and $[g_0,g_1,\dots,g_{n+1}] = [[g_0,g_1,\dots,g_n],g_{n+1}]$; repeated commutators are given by $[g,_0 h] = g$ and $[g,_{n+1} h] = [[g,_n h],h]$. The *descending central series* is defined by $G^0 = G$ and $G^{n+1} = [G,G^n]$; the *ascending central series* is given by $Z_0(G) = \{1\}$ and $Z_{n+1}(G) = \{g \in G : [g,G] \le Z_n(G)\}$. A group G is *nilpotent* of *class* n if $G^n = \{1\}$ or, equivalently, $G = Z_n(G)$; it is *locally nilpotent* if any finitely generated subgroup is nilpotent, and *uniformly locally nilpotent* if any finitely generated subgroup is nilpotent and the nilpotency class depends only on the number of generators. The series of *iterated centralizers* of some subset A of G is defined inductively via $C_G^0(A) = \{1\}$ and $C_G^{n+1}(A) = \{g \in \bigcap_{i\le n} N_G(C_G^i(A)) : [g,A] \subseteq C_G^n(A)\}$. We shall need the following facts, which can be found in [9]:

Fact 1.1 *[2] Suppose $K \le H \le G$, and $C_G(K^i) = C_G(H^i)$ for all $i < j$. Then $C_G^j(K) = C_G^j(H)$.*

Fact 1.2 *(T. Yen) A locally nilpotent $\mathcal{M}_c$-group is soluble.*

Fact 1.3 *[5, 4] The bounded left Engel elements in a soluble $\mathcal{M}_c$-group form the Fitting subgroup, which is nilpotent.*

Lemma 1.4 *Let G be a group, X a G-invariant subset, H a subgroup of G such that no proper subgroup of H is self-normalizing in H. Suppose K is a subgroup of G intersecting H in I such that $H \cap X \not\subseteq K$. Then there is $h \in N_{H\cap X}(I \cap X) - I$.*

Proof: If the whole of $H \cap X$ normalizes $I \cap X$, we are done. Otherwise $N_H(I \cap X)$ is a proper subgroup of H, so $N_H(N_H(I \cap X)) > N_H(I \cap X)$ and there is $g \in H$ which moves $I \cap X$, but fixes $N_H(I \cap X)$. As g must also stabilize $N_{H\cap X}(I \cap X)$ by G-invariance of X, we get $N_{H\cap X}(I \cap X) \supset I \cap X$. □

Lemma 1.5 *Let G be a group with nilpotent subgroups S and T of nilpotency class m and n, respectively. If $C_G^i(S) = C_G^i(T)$ for all $i \le m$, then $\langle S,T\rangle$ is nilpotent of class at most $m + n + 1$.*

Proof: Consider a sequence $I = (x_0, x_1, \ldots)$ of elements in $S \cup T$, and put $c_0 = x_0$, and $c_{i+1} = [c_i, x_{i+1}]$. If the first $n+1$ elements in I are from T, then $c_{n+1} = 1$. Otherwise, since $S \le C^m_G(S)$ and both S and T normalize $C^i_G(S) = C^i_G(T)$ for all $i \le n$, we obtain $c_{n+1} \in C^m_G(S)$. But now clearly $c_{m+n+1} \in C^0_G(S) = C^0_G(T) = \{1\}$, as the upper index goes down by at least one every time we take a commutator by an element in S or in T. But every commutator of length $m+n+1$ in $\langle S, T\rangle$ is a product of commutators of the above form, of length at least $m+n+1$, and thus trivial. $\square$

Definition 1.1 The *n-th Engel identity* is the condition $[x,_n y] = 1$.
An element $g \in G$ is *right Engel* if $[g,_n x] = 1$ for all x in G, where n may depend on x. If n can be chosen independently of x, then g is called *right n-Engel*, or *bounded right Engel*.
An element $g \in G$ is *left Engel* if $[x,_n g] = 1$ for all x in G, where again n may depend on x. If n can be chosen independently of x, then g is called *left n-Engel*, or *bounded left Engel*.
An element is *Engel* if it is left or right Engel. It is *bounded Engel* if it is bounded left or bounded right Engel.

Note that if g^{-1} is right n-Engel and x is in G, then $[x,_{n+1} g] = [g^{-x} g,_n g] = [g^{-x},_n g]^g = 1$, and so g is left $(n+1)$-Engel.

2 The Main Theorem

Theorem 2.1 *Let G be an $\mathcal{M}_c$-group generated by bounded Engel elements. Then G is nilpotent.*

Proof: Suppose not, and let G be a counter-example. If we denote the set of bounded left Engel elements by E, then E is invariant under conjugation by elements in G; since the inverse of a bounded right element is bounded left Engel, E generates G. By Fact 1.2 every locally nilpotent subgroup of G is soluble, and by Fact 1.3 every soluble subgroup of G generated by elements in E is nilpotent. It follows that maximal nilpotent subgroups generated by subsets of E exist and are maximally soluble (generated by subsets in E). In

particular, if S is such a group, then $N_E(S) \subseteq S$. Let n be the nilpotency class of S.

As E generates G, which is not nilpotent, it follows that there must be another maximal nilpotent subgroup $T \neq S$ of G generated by elements of E. Put $I := \langle S \cap T \cap E\rangle$. By the chain condition on centralizers, we may choose T such that $C_G(I^i)$ is minimal possible for all $i = 0, 1, 2, \ldots, n$ (in that order), subject to $T \neq S$.

Consider any $s \in N_{S\cap E}(I) - I$ (which exists by Lemma 1.4, since $I < S$ and $S \cap E$ generates S). For a similar reason, there is $t \in N_{T\cap E}(I) - I$, and $[t, s, s, \ldots, s] = 1$ by the Engel condition. Hence there is a maximal commutator $u = [t, s, \ldots, s]$ such that $u \notin S$, but $[u, s] \in S$. It follows that $s^u \in S$. Now either $u = t \in E$, or $u = v^{-1}s$ for some conjugate $v \in E - S$ of s; since $N_E(S) \subseteq S$, either case implies $S \neq S^u$, and we may replace T by $T_0 := S^{u^{-1}}$. Note that u normalizes I, so $\langle I, s\rangle \le S \cap T_0$. It follows that for any finite tuple $\bar{s}$ of elements in $N_{S\cap E}(I)$ we may replace T by some $T_0 \neq S$ with $\langle I, \bar{s}\rangle \le S \cap T_0$. If $J := S \cap T_0 \cap E$, then clearly $\langle J\rangle \ge \langle I, \bar{s}\rangle$; the chain condition on centralizers and minimality of $C_G(I^i)$ for all $i \le n$ now imply $C_G(I^i) = C_G(\langle N_{S\cap E}(I)\rangle^i)$ for all $i \le n$.

Note that the above argument also shows that we may assume that T is a conjugate of S, say $T = S^g$. Conjugating by g^{-1}, we can exchange the rôles of S and T; in particular we see that $C_G(I^i)$ is also minimal for $i = 0, 1, 2, \ldots, n$ when we fix T and vary S, and T has nilpotency class at most n. Furthermore, by symmetry we obtain $C_G(I^i) = C_G(\langle N_{T\cap E}(I)\rangle^i)$ for all $i \le n$.

Now let X be a subset of $S \cap E$ such that $C_G(\langle X\rangle^i) = C_G(I^i)$ for all $i \le n$. As $X \subseteq S$, it generates a nilpotent group of class at most n; by Fact 1.1,

$$C^i_G(\langle X\rangle) = C^i_G(I) = C^i_G(\langle N_{T\cap E}(I)\rangle)$$

for all $i \le n$. Then $\langle X, N_{T\cap E}(I)\rangle$ is nilpotent by Lemma 1.5, and is contained in a maximal nilpotent subgroup $T_0 \le G$ which is generated by elements of E. Since $N_{T\cap E}(I) \not\le S$, clearly $T_0 \neq S$. Now $J := \langle S \cap T_0 \cap E\rangle$ contains I, so $C_G(J^i)$ is minimal possible as well for all $i \le n$, and we see as above that $C_G(I^i) = C_G(J^i) = C_G(\langle N_{S\cap E}(J)\rangle^i)$ for all $i \le n$. It follows from Lemma 1.1 that if we define $I_0 := I$, $I_{\alpha+1} = \langle N_{S\cap E}(I_\alpha)\rangle$ and $I_\lambda = \bigcup_{\alpha<\lambda} I_\alpha$ for limit ordinal λ, we shall eventually exhaust $S \cap E$ and obtain $S = \langle S \cap E \rangle = I_\kappa$ for κ big enough. But a transfinite induction on α (using the chain condition on centralizers for the limit step) yields $C_G(I^i_\alpha) = C_G(I^i)$ for all $i \leq n$ and all $\alpha \leq \kappa$; in particular $C_G(I^i) = C_G(I^i_\kappa) = C_G(S^i)$ for all $i \leq n$. By symmetry $C_G(I^i) = C_G(T^i)$ for all $i \leq n$; Lemma 1.1 then yields,

$$C^i_G(S) = C^i_G(I) = C^i_G(T)$$

for all $i \leq n$, and $\langle S, T \rangle$ is nilpotent by Lemma 1.5. Since $S \not\leq T$, this contradicts the maximality of S. □

Corollary 2.2 *The set of bounded left Engel elements in an $\mathcal{M}_c$-group equals the Fitting subgroup.*

Proof: Apply Theorem 2.1 to the group G generated by the bounded left Engel elements (which are also left Engel in G, and form a G-invariant set).

□

Problem 2.1 In an $\mathcal{M}_c$-group, is the set of left Engel elements equal to the Hirsch-Plotkin radical?

Problem 2.2 A *commutator condition* (in two variables) is a non-trivial iterated commutator $[x, y, z_1, \ldots, z_n]$ with $z_i \in \{x, y\}$ for all $i = 1, \ldots, n$. If an $\mathcal{M}_c$-group satisfies $w(x, y) = 1$, is it nilpotent? What if the group is substable?

Problem 2.3 Characterize the sets of right and of bounded right Engel elements in an $\mathcal{M}_c$-group, or in a substable group.

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MANY-SORTED ABELIAN ALGEBRAS IN CONGRUENCT MODULAR VARIETIES

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1 Introduction

In [Wol96], we characterized many-sorted Abelian algebras in congruence modular varieties with help of a many-sorted version of the Day–Kiss–Ring (cf. [DK87]). In this paper we are giving a characterization in the same way C. Herrmann [Her79] did for the one-sorted case.

We start with a short introduction to the basics of many-sorted universal algebra (cf., for example, [MT92]). A **signature** of many-sorted algebras is a triple $(S, \mathcal{F}, \sigma)$, where S is a set (of sorts), $\mathcal{F}$ an S-sorted set (i.e., an S-indexed family $(\mathcal{F}_s \mid s \in S)$ of sets) of operation symbols and $\sigma: \mathcal{F} \to S^* \times S$ an arity function. We write $f: s^1 \times \ldots \times s^m \to s$ if $\sigma(f) = ((s^1, s^2, \ldots, s^m), s)$. If $m = 0$, then f is called **constant operation symbol**. Given a signature $(S, \mathcal{F}, \sigma)$, a **many sorted algebra** (or, more exactly, **S-sorted algebra**) $\mathbf{A}$ of this signature is a pair $(A; F)$ consisting of an S-sorted set A, called the **universe** of $\mathbf{A}$, and of an S-sorted set $F = (f^A \mid f \in \mathcal{F}_s)_{s \in S}$ of fundamental operations which are realizations of the operation symbols in $\mathcal{F}$. In this paper we assume that every set A_s is nonempty. Algebras of same signature are called **similar**.

A **many-sorted variety** (or **S-sorted variety**) of signature $(S, \mathcal{F}, \sigma)$ is a class of algebras of this signature closed under the formation of direct products, subalgebras, and homomorphic images. $\mathsf{V}(\mathcal{K})$ is the **variety generated** by $\mathcal{K}$, i.e., the least variety containing $\mathcal{K}$. A many sorted variety $\mathcal{V}$ is called **congruence modular** if the congruence lattice of every algebra $\mathbf{A} \in \mathcal{V}$ is a modular lattice.

Two many-sorted algebras $\mathbf{A}$ and $\mathbf{B}$ with same universes are **polynomial equivalent** if they have the same polynomial functions.

Many other basic definitions of one-sorted algebras, like subalgebras, congruence relations, and homomorphisms, can be transferred to many-sorted algebras. An example for many-sorted algebras we will use later are the many sorted counterparts of rings and modules.

Example 1.1 Let S be a set. A **ringoid** is an $S \times S$-sorted algebra

$$\begin{aligned} \mathbf{R} \ := \ ((R_{s,t} \mid s,t \in S); \ & ((+,-,0,\circ,1)_{s,s} \mid s \in S) \cup \\ & ((+,-,0)_{s,t} \mid s,t \in S, s \neq t) \cup \\ & (\circ_{s,t,u} \mid s,t,u \in S, s \neq t \text{ or } t \neq u)) \end{aligned}$$

where

$$\mathbf{R}_{s,s} := (R_{s,s}; (+,-,0,\circ,1)_{s,s})$$

is a unitary ring,

$$\mathbf{R}_{s,t} := (R_{s,t}; (+,-,0)_{s,t})$$

is an Abelian group, and $\circ_{s,t,u} : R_{s,t} \times R_{t,u} \to R_{s,u}$ is bilinear, for all $s,t,u \in S$. The latter is

$$\begin{aligned} & p \circ_{s,t,u} (q +_{t,u} r) = (p \circ_{s,t,u} q) +_{s,u} (p \circ_{s,t,u} r) \quad \text{and} \\ & (q +_{s,t} r) \circ_{s,t,u} p = (q \circ_{s,t,u} p) +_{s,u} (r \circ_{s,t,u} p). \end{aligned}$$

A **many sorted module** over a given $S \times S$-sorted ringoid $\mathbf{R}$ is an S-sorted algebra

$$\mathbf{M} \ := \ ((M_s \mid s \in S); ((+,-,0)_s \mid s \in S) \cup (R_{s,t} \mid s,t \in S))$$

where, for all $s,t \in S$, $(M_s; (+,-,0)_s, R_{s,s})$ is a one sorted unitary module over $\mathbf{R}_{s,s}$, and $\mathbf{R}_{s,t}$ is a group of homomorphisms from $(M_t; (+,-,0)_t)$ to $(M_s; (+,-,0)_s)$, such that the 0 of $\mathbf{R}_{s,t}$ behaves as zero morphism.

Ringoids, as natural extensions of rings to the many sorted case, were introduced by B. Mitchell in [Mit72] as small additive categories. Similarily, every many sorted module can be defined as an additive functor from a ringoid to the category of Abelian groups, c.f. [Row96].

2 Restrictions

In [Wol96] we introduced the restriction of an S-sorted algebra $\mathbf{A}$ to a sort $s \in S$ and showed the results needed here. The universe is the set A_s while the fundamental operations are special term functions of the algebra $\mathbf{A}$ over the S-sorted variable set X with $X_s = \{x_s^1, x_s^2, \dots\}$ and $X_t = \emptyset, t \neq s$:

Definition 2.1 Let $\mathbf{A}$ be an algebra of signature $(S, \mathcal{F}, \sigma)$. Then for each $s \in S$, the one-sorted algebra

$$\boxed{\mathbf{A}_s := (A_s; (u^{\mathbf{A}} \in T(\mathbf{A}) \mid u \in T_s(X), u : s \times \dots \times s \to s))}$$

is called the **restriction of the algebra A to the sort** s. If $\mathcal{K}$ is a class of algebras of signature $(S, \mathcal{F}, \sigma)$, we define

$$\mathcal{K}_s := \{\mathbf{A}_s \mid \mathbf{A} \in \mathcal{K}\}.$$

Note that the restrictions of two similar many-sorted algebras to the same sort have the same one-sorted type. Therefore, the restriction of a class of similar many-sorted algebras gives a class of similar one-sorted algebras.

Example 2.2 Let $\mathbf{M}$ be a many-sorted module over the ringoid $\mathbf{R}$. Then the restriction $\mathbf{M}_s$ to the sort s is term equivalent to the one-sorted module $(M_s; (+, -, 0)_s, R_{s,s})$.

A strong connection between the variety generated by a class of similar algebras and the variety generated by the restriction to a sort can be obtained.

Theorem 2.3 *Let $\mathbf{A}$ be an S-sorted algebra. Then for all $s \in S$ we have*

$$\mathsf{V}(\mathcal{K})_s = \mathsf{V}(\mathcal{K}_s).$$

In addition, we have:

Theorem 2.4 *Let $\mathcal{V}$ be an S-sorted variety. Then the following are equivalent:*

a) $\mathcal{V}$ is congruence modular.

b) For every many sorted algebra $\mathbf{A} \in \mathcal{V}$ and every sort $s \in S$, $\mathsf{V}(\mathbf{A}_s)$ is congruence modular.

3 Abelian Algebras

In this section we characterize many sorted Abelian algebras which belong to a congruence modular variety.

Definition 3.1 A **many sorted Abelian algebra** is an algebra $\mathbf{A}$ with the property that $\{(a, a) \mid a \in A_s\}$ is a class of δ_s for all $s \in S$,

where δ is the congruence

$$\delta := Cg^{\mathbf{A} \times \mathbf{A}}(\{((a, a), (b, b)) \mid a, b \in A\}).$$

If $\mathbf{A}$ is an Abelian algebra in a congruence modular variety, then the congruence δ is a complement of the kernels η^1 and η^2 of the projections p^1 and p^2 in the congruence lattice of $\mathbf{A} \times \mathbf{A}$.

The basis of this section is provided by the result of C. Herrmann [Her79] that in a one sorted congruence modular variety every Abelian algebra $\mathbf{A}$ is polynomially equivalent to a module over a suitable ring (see also [FM87] or [Gum83]). With an arbitrary element $0 \in A$, an Abelian group $\mathbf{A}^+ = (A; +, -, 0)$ is constructed, where $+$ and $-$ are polynomial functions of $\mathbf{A}$ which are derived from the unique **difference term of the congruence** modular variety. For the algebra $\mathbf{A}$, we have $d(a, b, c) = a - b + c$. Another

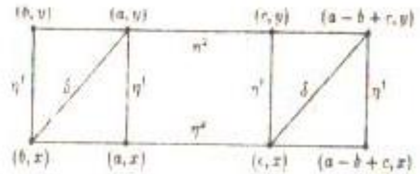

Figure 1: Definition of $a - b + c$

way to construct $a - b + c$ with help of the congruences δ, η^1 and η^2 is shown in Figure 1, where x is an arbitrary element of $\mathbf{A}$. Then all polynomial functions are **affine** with respect to $\mathbf{A}^+$, and $\mathbf{A}$ is polynomially equivalent to the module ${}_P\mathbf{A}^+ = (A; +, -, 0, P)$ over $\mathbf{P} = (\{p(x) \in P(\mathbf{A}) \mid p(0) = 0\}; +, -, 0, \circ, id_A)$ which is a unitary ring of endomorphisms of $\mathbf{A}^+$.

Now, if an S sorted algebra $\mathbf{A}$ is Abelian and belongs to a congruence modular variety, then Theorem 2.4 and the fact that every congruence of $\mathbf{A}$

consist of a family of congruences of $\mathbf{A}$, ensure the same properties for the restriction $\mathbf{A}_s$ for each $s \in S$. Thus we can construct Abelian groups $\mathbf{A}_s^+$ for every sort and we have:

Lemma 3.2 *Let* $\mathbf{A}$ *be an* S*-sorted algebra, let* $\mathsf{V}(\mathbf{A})$ *be congruence modular, let* $s \in S$*, and let* 0_s *be an element of* A*. Then*

$$ {}_{P_s}\mathbf{A}_s^+ = (A_s; +_s, -_s, 0_s, P_s) $$

is a unitary module over the ring

$$ \mathbf{P}_s = (\{p(x) \in P(\mathbf{A}_s) \mid p(0_s) = 0_s\}; +, -, 0_s, \circ, id_{A_s}), $$

and $\mathbf{A}$ *is polynomially equivalent to* $(A_s; +_s, -_s, 0_s, P_s)$.

This is not enough to characterize a many sorted Abelian algebra $\mathbf{A}$ in a congruence modular variety in the same way as we can characterize one-sorted algebras, because there might exist unary polynomial functions of $\mathbf{A}$ with domain and range A_s mapping 0_s to 0_s, which are not polynomial functions of $\mathbf{A}_s$. Let us proceed by showing that all polynomial operations are affine.

Definition 3.3 An operation $f : A^1 \times \ldots \times A^m \to A$ is called **affine** with respect to the Abelian groups $(A^1, +, -, 0), \ldots, (A^m, +, -, 0), (A, +, -, 0)$ if, for all $a^1, b^1, c^1 \in A^1, \ldots, a^m, b^m, c^m \in A^m$,

$$
\begin{aligned}
& f(a^1 - b^1 + c^1, \ldots, a^m - b^m + c^m) \\
&= f(a^1, \ldots, a^m) - f(b^1, \ldots, b^m) + f(c^1, \ldots, c^m)
\end{aligned}
$$

holds.

Lemma 3.4 *Let* $\mathbf{A}$ *be an* S*-sorted Abelian algebra in a congruence modular variety. Then every polynomial function of* $\mathbf{A}$ *is affine.*

Proof: Figure 1 illustrates the operation $a -_s b +_s c$ in $\mathbf{A}_s^+$ if we replace η^1, η^2, and δ with η^1_s, η^1_s, and δ_s. Since every polynomial function p respects this diagram, we obtain the same diagram for $p(a^1, \ldots, a^m)$, $p(b^1, \ldots, b^m)$, $p(c^1, \ldots, c^m)$ and $p(a^1 - b^1 + c^1, \ldots, a^m - b^m + c^m)$, implying

$$ p(a^1 - b^1 + c^1, \ldots, a^m - b^m + c^m) = p(a^1, \ldots, a^m) - p(b^1, \ldots, b^m) + p(c^1, \ldots, c^m). $$

Hence p is affine.

Using this, we can characterize many-sorted Abelian algebras in congruence modular varieties.

Theorem 3.5 *Let* $\mathbf{A}$ *be an* S*-sorted Abelian algebra such that* $\mathsf{V}(\mathbf{A})$ *is congruence modular. Then* $\mathbf{A}$ *is polynomially equivalent to an* S*-sorted module over a ringoid.*

Proof: We choose elements $0_s \in A_s$ for all $s \in S$. Now it is more advantageous to examine $\mathbf{A}_A$ (i.e., the algebra derived from $\mathbf{A}$ by adding all elements of A as constant operations) instead of $\mathbf{A}$. Notice that $\mathbf{A}_A$ is also an Abelian algebra in a congruence modular variety and polynomially equivalent to $\mathbf{A}$. Thus, like in Lemma 3.2, $(\mathbf{A}_A)_s$ is polynomially equivalent to a module over the ring

$$ \mathbf{R}_{s,s} := (\{r(x) \in P((\mathbf{A}_A)_s) \mid r(0_s) = 0_s\}; +_s, -_s, 0_s, \circ, id_{A_s}) $$

for every $s \in S$. Since

$$ \{r(x) \in P((\mathbf{A}_A)_s) \mid r(0_s) = 0_s\} = \{r(x) \in P(\mathbf{A}) \mid r : A_s \to A_s, r(0_s) = 0_s\}, $$

the set of all unary polynomial functions of $\mathbf{A}$ with domain and range A_s, mapping 0_s to 0_s, form a ring. Moreover, for all $s, t \in S$, we can define an Abelian group structure on the polynomial functions

$$ R_{s,t} = \{r \in P(\mathbf{A}) \mid r : A_t \to A_s, r(0_t) = 0_s\} $$

by

$$
\begin{aligned}
(r^1 +_{s,t} r^2)(a) &:= r^1(a) +_s r^2(a), \\
(-_{s,t} r)(a) &:= r(-_t a) \text{ and} \\
0_{s,t} &:= 0_s,
\end{aligned}
$$

where $a \in A_t$, and $+_s$ and $-_s$ are the group operations of $\mathbf{A}_s^+$. The composition $\circ_{s,t,u} : R_{s,t} \times R_{t,u} \to R_{s,u}$ is bilinear since the polynomial functions are affine:

$$
\begin{aligned}
(p \circ_{s,t,u} (q +_{t,u} r))(a) &= p((q +_{t,u} r)(a)) \\
&= p(q(a) -_t 0_t +_t r(a)) \\
&= p(q(a)) -_s p(0_t) +_s p(r(a)) \\
&= ((p \circ_{s,t,u} q) +_{s,u} (p \circ_{s,t,u} r))(a).
\end{aligned}
$$

This proves that

$$\mathbf{R} := ((R_{s,t} \mid s,t \in S); ((+,-,0,\circ,1)_s \mid s \in S) \cup ((+,-,0)_{s,t} \mid s,t \in S, s \neq t) \cup (\circ_{s,t,u} \mid s,t,u \in S, s \neq t \text{ or } t \neq u))$$

is a ringoid and, as all polynomial functions in $R_{s,t}$ are group homomorphisms from $\mathbf{A}_t^+$ to $\mathbf{A}_s^+$, that

$$_R\mathbf{A}^+ := ((A_s \mid s \in S); ((+,-,0)_s \mid s \in S) \cup (R_{s,t} \mid s,t \in S))$$

is a many-sorted module over $\mathbf{R}$. Now, every polynomial function

$$p : A_{s^1} \times \ldots \times A_{s^m} \to A_s$$

of $\mathbf{A}$ is of the form

$$p(x^1,\ldots,x^m) = p(0_{s^1},\ldots,0_{s^m}) + r^1(x^1) + \ldots + r^m(x^m)$$

with

$$r^i(x^i) = p(0_{s^1},\ldots,0_{s^{i-1}},x^i,0_{s^{i+1}},\ldots,0_{s^m}) - p(0_{s^1},\ldots,0_{s^m}) \in R_{s,s^i},$$

because p is affine. This shows that $\mathbf{A}$ and $_R\mathbf{A}^+$ are polynomially equivalent. □

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Abstracts

R.D. Aref'ev. *On a property of monotomousness of weakly o-minimal models.*

A property of monotonousness of unary functions, defined in weakly o-minimal models, is proved.

S.B. Baizhanov. *Two theorems on o-minimal theories.*

A criteria of existence of essential expantion of models of o-minimai theories is given in terms of partial unar functions. The second result is connected with the following problem: when the expansion of o-minimal model by convex predicate is weakly o-minimal.

R. El Bashir, T. Kepka. *Equationally Complete Classes of Semirings.*

All equationally complete classes of semirings are found.

S.A. Berezin. *On subalgebras of iterational algebras of recursive functions.*

Some subalgebras in algebras of primitive- and recursive unary functions with operations of composition and recursion are considered.

I. Chajda, J. Zednik. *On n-permutability of varieties having p-determined congruences.*

A connection between a property of a variety to have p determined congruences (p-determined tolerances) and its n-permutability is considered.

K. Denecke, D. Welke. *Solid varieties of partial algebras.*

Partial algebras belong to the basic mathematical structures in computer science. We transfer the concept of a hyperidentity from total to partial algebras. This concept is based on strong identities. We develop a hyperequational theory for partial algebras and the corresponding fragment of second order logic.

A.A. Ivanov. *Finite covers with linking equivalence relations.*

A criteria of the splitness strong splitness of the cover of some structures is given.

B.Sh. Kulpeshov. *Some properties of $\aleph_0$-categorical weakly o-minimal theories.*

The structure of the set of realizations of a 1-type in $\aleph_0$-categorical weakly o-minimal theories are investigated. Indiscernibility of the set of realizations of any non-algebraic 1-type in such a theory is proved. It is shown that in $\aleph_0$-categorical weakly o-minimal theories of convexity rank 1 the 2-indiscernibility implies the indiscernibility.

S.I. Mardaev. *Modal Π-schemes.*

A wide class of Kripke's models with definable least solutions of modal positive Π-schemes is shown.

Ya.L. Mordvinov. *On simple skeletons of congruence-modular varieties.*

A description of congruence-modular varieties with linear countable skeletons of epimorphisms is given.

Sh. Mynbaeva. *Some questions of weakly o-minimal binary theories.*

Several properties of weakly o-minimal binary theories, connected with independance and indescernibility of subsets of models and description of definable functions, are proved.

A.G. Pinus. *On algebras of conditionally termal functions.*

We investigate a connection between properties of a universal algebra and an algebra of conditionally termal functions of this algebra.

A.G. Pinus. *Inner isomorphisms and conditionally rationally equivalence to unars and to fields.*

On the language of inner isomorphisms the author gives a characterization of universal algebras that are conditionally rationally equivalent to unars and to fiealds.

K.N. Ponomaryov. *Covers of algebraic varieties.*

The author gives a sketch of his approach to branched covers of algebraic varieties.

A.M. Popova. *The description of unit groups of irredusible matrix rings.*

In this research the problem of description of unit group of irredusible matrix ring over the field of rational numbers is solves. The case, when linear shell of the ring over the field of rational numbers is isomorphic to complete matrix algebra over the division ring, is completely researched.

In accordance with other author's results it gives us ability to describe the unit groups of any finitely generated matrix rings over the field of rational numbers and solve partial case of the L.Fuchs' problem in the class of such rings.

Branimir Šešelja. *On the congruence intersection property.*

Congruence Intersection Property (CIP) and its variations establish connections between congruences on arbitrary (different) subalgebras of an algebra. When formulated in lattice theoretic terms, CIP introduces particular duality between subalgebras and congruences in the lattice of weak congruences of an algebra. The aim of this paper is to present some algebraic aspects of the mentioned duality. Such properties of the CIP turn out to be useful in characterizations of some classes of algebras and varieties. Various conditions under which an algebra and some varieties or classes of algebras satisfies the CIP, are also given.

S.V. Sudoplatov. *Trigonometries with functions Sin and Cos.*

The axiomatization of trigonometries allowing to define the main trigonometrical functions is considered.

Andreja Tepavčević. *On representation of lattices by weak congruences and weak tolerances.*

The problem of representation of lattices by weak congruences with a fixed element representing the diagonal relation is solved here for a wide class of lattices. It turns out that the representation can be given in a similar way by weak tolerances and by weakly reflexive compatible relations.

J. Ušan. *n-groups, $n \geq 2$, as varieties of type $< n, n-1, n-2 >$.*

In this article n-groups (:[1]; 2.1.1) for any $n \geq 2$ are described as algebras of the type $(n, n-1, n-2)$ in which the $(n-2)$-ary operation is described as a generalization of a nullary operation - fixing the neutral element of a groupoid (:[10]; 2.2.1), and the $(n-1)$-ary operation represents a generalization of the inversing operation in a group (:[1]; 3.1-3.4). Among the results of the paper is the following proposition. Let $n \geq 2$ and let (Q, A) be an n-groupoid. Then: (Q, A) is an n-group (:[1]; 2.1.1) iff there are mappings $^{-1}$ and e respectively of the sets Q^{n-1} and Q^{n-2} into the set Q such that the following laws hold in the algebra $(Q, \{A, ^{-1}, e\})$ [of the type $(n, n-1, n-2)$]: (i) $A(x_1^{n-2}, A(x_{n-1}^{2n-2}), x_{2n-1}) = A(x_1^{n-1}, A(x_n^{2n-1}))$; (ii) $A(e(a_1^{n-2}), a_1^{n-2}, x) = x$; and (iii) $A((a_1^{n-2}, a)^{-1}, a_1^{n-2}, a) = e(a_1^{n-2})$. In addition: If the law (i) holds in the n-groupoid (Q, A), $n \geq 2$, then there is at most one e and at most one $^{-1}$ such that the laws (ii) and (iii) also hold. /For $n = 2$ $e(a_1^{n-2})$ $[= e(\emptyset)$; 1] is a left neutral element of the groupoid (Q, A), $(a_1^{n-2}, a)^{-1}$ $[= a^{-1}]$ is a left inverse element of the element a with respect to $e(\emptyset)$, and the law (i) reduces to the law $A(A(x_1^2), x_3) = A(x_1, A(x_2^3))$. For $n = 2$, it is, hence, one of the well known characterizations of a group, with a minimality property./

V.V. Verbovsky. *On formula depth of weakly o-minimal structures.*

In the paper of D.Macpherson, D.Marker, C.Steinhorn (1993) some basic properties of weakly o-minimal structures had been investigated. In this work we introduce new notions "finite depth" and "strong monotonicity" with the help of which we define new necessary conditions of weak o-minimality of elementary theory of a weakly o-minimal structure. We adduce in the second part examples of weakly o-minimal structures without weakly o-minimal theories.

F. Wagner. *Engel elements in groups with the minimal condition on centralizers.*

A group with the chain condition on centralizers which is generated by Engel elements is nilpotent.

Ch. Wolf. *Many-sorted abelian algebras in congruent modular varieties.*

It is given a criteria of abelinness for many-sorted algebras, like a Hermann's criteri of abelianness of one-sorted algebras.

Contents

Александр Георгиевич Пинус,
Константин Николаевич Пономарев

АЛГЕБРА И ТЕОРИЯ МОДЕЛЕЙ

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