ON THE UNIVERSALITY OF CS$\mathfrak{X}$ AND $\mathfrak{X}$T-GROUPS

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A subgroup H of a given group G is called *malnormal* if for every element $x \in G \setminus H$, we have $H \cap H^x = \{1\}$. A group G is called *conjugate separable abelian* (CSA) if all maximal abelian subgroups of G are malnormal. The class of CSA-groups is broad and plays a significant role in the study of residually free groups, universal theory of non-abelian free groups, limit groups, exponential groups and equational domains in algebraic geometry over groups (see [1], [2], [10] , and [11]). Another class of groups with very close connections to CSA-groups is the class of CT-groups (*commutative transitive groups*). A group is CT if commutativity is a transitive relation on the set of its non-identity elements. Despite this simple definition, the class of CT groups is also central to the study of residually free groups. Every CSA-group is CT but the converse is not true. B. Baumslag proved that in the presence of residual freeness, both properties are equivalent.

In recent decades, considerable effort has been devoted to the study of these classes and their generalizations. A generalization of CT-groups is introduced in [3] to extend the above mentioned theorem of B. Baumslag. Many interesting relations between CSA- and CT-groups are presented in [5], along with an extensive review of the existing literature.

I proposed an idea in [14] for generalizing the notions of CSA- and CT-groups: Suppose that $\mathfrak{X}$ is a class of groups. A group G is an $\mathfrak{X}$T-group if and only if, for any two $\mathfrak{X}$-subgroups A and B of G, the assumption $A \cap B \neq \{1\}$ implies that $\langle A, B\rangle$ is also an $\mathfrak{X}$-subgroup of G. Similarly, a group G is a CS$\mathfrak{X}$-group if all of its maximal $\mathfrak{X}$-subgroups are malnormal. In [14], I examined the special case where $\mathfrak{X} = \mathfrak{N}_k$, the variety of nilpotent groups of nilpotency class not exceeding k, which contains CSA- and CT-groups as particular instances. Hence, [14] addresses the study of general relations among the classes of CS$\mathfrak{X}$- and $\mathfrak{X}$T-groups, their characterizations, constructions, and universal axiomatization, as well as their connections to residual A-free groups, in the case when $\mathfrak{X}$ is the variety of nilpotent groups of class at most k. Furthermore, many previous results are shown to remain valid for CS$\mathfrak{X}$ and $\mathfrak{X}$T-groups when $\mathfrak{X}$ is the variety $\mathfrak{N}_k$. As an application, the ideas of [14] are used in [12] to introduce a large class of groups which are *equational domain* in the sense of algebraic geometry over groups (see [1] for definitions).

Recently, with my student, Omar Al-Raisi, we developed the theory of CS$\mathfrak{X}$- and $\mathfrak{X}$T-groups for arbitrary varieties of groups in [13] and we showed that many results from [14] remain true in this general framework, while some are failed. For example, we are not sure that these classes are universal still or not. In this talk, I consider a situation where both classes can be axiomatized by a set of universal axioms. Here is a review of our main results:

The classes CT and CSA are universal; they can be axiomatized by universal sentences in the first order language of groups (see [5]). The same is true for the classes $\mathfrak{X}$T and CS$\mathfrak{X}$ when $\mathfrak{X}$ is the variety of nilpotent groups of class at most k (see [14]). We generalize this to a large class of varieties; finitely based varieties $\mathfrak{X}$ such that the 2-generator relatively free group of $\mathfrak{X}$ is finitely presented. Note that this includes every finitely based locally finite variety as well as every virtually nilpotent variety. For every variety $\mathfrak{X}$, the 2-generator relatively free element of $\mathfrak{X}$ will be denoted by $F_2(\mathfrak{X})$.

Theorem. Suppose $\mathfrak{X}$ is a finitely based variety such that $F_2(\mathfrak{X})$ is finitely presented. Then, the class $\mathfrak{X}$T is universal. Further, if CS$\mathfrak{X} \subseteq \mathfrak{X}$T, then CS$\mathfrak{X}$ is universal as well.

There are many examples of varieties satisfying the assumptions of the above theorem among which are the varieties of nilpotent groups of class at most k, every finitely based locally finite variety (equivalently, every locally finite variety with finite axiomatic rank) and in general, every virtually finite nilpotent variety. However, we are not sure about the existence of any more non-trivial examples. An old question of A. Olshanskii about the existence of a non-trivial example of finitely presented relatively free group which is not virtually nilpotent is still unsolved (see Problem 11.73 in [7]). One can use the above theorem to produce infinitely many examples of $\mathfrak{X}$T- and CS$\mathfrak{X}$-groups.

Corollary. Suppose $\mathfrak{X}$ is a finitely based variety such that the relatively free element $F_2(\mathfrak{X})$ is finitely presented. Then, any ultra-product of $\mathfrak{X}$T-groups is $\mathfrak{X}$T. If further, CS$\mathfrak{X} \subseteq \mathfrak{X}$T, then every ultra-product of CS$\mathfrak{X}$-

groups is CS$\mathfrak{X}$.

As an example, if we have two CS$\mathfrak{X}$-groups A and B without involutions, then for any set I and any ultra-filter $\mathcal{U}$ over I, the ultra-power $(A * B)^I/\mathcal{U}$ is a new CS$\mathfrak{X}$-group. Consequently, we have an infinite supply of new CS$\mathfrak{X}$-groups (and hence, equational domains).

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