Isomorphism criterion for $GBS(n,1)$ groups with non-trivial center

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A finitely generated group G_n that acts on a tree T such that all edge stabilizers are infinite cyclic groups and all vertex stabilizers are free abelian groups of rank n will be called a *generalized Baumslag–Solitar group of type (n,1)* ($GBS(n,1)$ group).

According to the Bass–Serre theorem, every $GBS(n,1)$ group G_n can be represented as $\pi_1(\mathbb{A})$ the fundamental group of a graph of groups $\mathbb{A}$ [1], where each edge group is infinite cyclic and each vertex group is a free abelian group of rank n. A $GBS(1,1)$ group is called a *generalized Baumslag–Solitar group* (GBS group). The isomorphism problem for $GBS(n,1)$ groups, even in the case $n=1$, has only been solved in some special cases and, in general, remains a difficult open problem. A survey of the known results can be found, for example, in [2].

In [3], a criterion for the existence of a nontrivial center in $GBS(n,1)$ groups was proposed. Furthermore, it turned out that every $GBS(n,1)$ group G_n with a nontrivial center can be given by the following presentation. If v is a vertex of A, let $v_1, v_2, \ldots, v_n$ denote generators of the corresponding vertex group. Let $\overline{A}$ be the graph obtained from A by identifying e and $\overline{e}$. A maximal subtree R of $\overline{A}$ yields the following presentation of the group $\pi_1(\mathbb{A}) \simeq G_n$:

$$\left\langle \begin{array}{l} v_1,\ldots,v_n \ (v \in V(A)), \\ t_e \ (e \in E(\overline{A}) \setminus E(R)) \\ \end{array} \middle| \begin{array}{l} v_1^{\lambda_e} = u_1^{\lambda_{\overline{e}}} \ (e=(u,v) \in E(R)), \\ v_i v_j = v_j v_i \ (1 \leqslant i \neq j \leqslant n, v \in V(A)), \\ t_e^{-1} v_1^{\lambda_e} t_e = u_1^{\lambda_{\overline{e}}} \ (e=(u,v) \in E(\overline{A}) \setminus E(R)) \end{array} \right\rangle,$$

where $\lambda_e, \lambda_{\overline{e}},\ e \in E(A)$ are nonzero integers.

Moreover, the subgroup $G = \langle v_1 \ (v \in V(A)),\ t_e \ (e \in E(\overline{A}) \setminus E(R)) \rangle$ of G_n is itself a GBS group. In [3], it is called *corresponding GBS subgroup* of the group G_n. The main result of [3] is a necessary condition for the isomorphism of $GBS(n,1)$ groups with nontrivial center: If G_n and H_n are two isomorphic $GBS(n,1)$ groups with nontrivial center and $n \geqslant 3$, then their corresponding GBS subgroups G and H are also isomorphic.

In this talk, we discuss why the isomorphism of corresponding GBS subgroups is not sufficient for the isomorphism of $GBS(n,1)$ groups with nontrivial center when $n \geqslant 3$, and give an additional condition which, provides a criterion for the isomorphism of $GBS(n, 1)$ groups with nontrivial center when $n \geqslant 3$.

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References

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