On sets of K_p in some finite groups

Anna Zabarina, Elena Fomina

1 Properties of the set K_p

Let G is a non-Abelian group. We denote:

$$K_p = \{g \in G \mid o(g) = p \wedge C(g) = \langle g \rangle\}, \qquad X_p = \{g \in G \mid o(g) = p\}.$$

The following properties have been proven [1].

- $a \in K_p \Rightarrow \forall g \, (a^g \in K_p)$.
- $a \in K_p \Rightarrow \langle a \rangle \setminus \{e\} \subset K_p$.
- $K_p \neq \varnothing \Rightarrow Z(G) = \{e\}$.

  $|G| = n$

- $K_p \neq \varnothing \Rightarrow n \vdots p \wedge n \not\vdots p^2 \wedge X_p = K_p$.
- $|K_p| \in \left\{0, \frac{n}{p}, \frac{2n}{p}, ..., \frac{(p-1)n}{p}\right\}$.

2 Sets K_p in groups S_n и A_n

The following results were obtained [1].

1. $p = 2$
    - $K_2(S_n) \neq \varnothing \Leftrightarrow n = 3, |K_2(S_3)| = \frac{|S_3|}{2}$;
    - $K_2(A_n) = \varnothing \Leftrightarrow n > 3$.
2. $p > 2$

- $S_n, n \geqslant 3$.

$$1)\ 0 \leqslant n - p \leqslant 1 \Rightarrow |K_p| = \frac{|S_n|}{p};$$
$$2)\ n - p > 1 \Rightarrow K_p = \varnothing.$$

- $A_n, n > 3$.

$$1)\ n-p \leqslant 1 \Rightarrow |K_p| = \tfrac{2|A_n|}{p};$$
$$2)\ n-p = 2 \Rightarrow |K_p| = \tfrac{|A_n|}{p};$$
$$3)\ n-p > 2 \Rightarrow K_p = \varnothing.$$

3 K_p in finite nilpotent groups

Theorem 1. *Let G is a nilpotent group, $|G| = n$, $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_s^{\alpha_s}$.*
Then $\forall i\, K_{p_i} = \varnothing$.

4 K_p in finite solvable groups

1) Let $\langle G, \cdot \rangle$ is a group, $|G| = pq$, where p, q are prime numbers and $p < q$. Then $|K_p| = (p-1)q$, $|K_q| = q-1$ [1].

2) Let $\langle G, \cdot \rangle$ is a metacyclic group, $Z(G) = e$, $|G| = p_1 p_2 ... p_s$, where $s \geqslant 3$, $p_i < p_{i+1}$. Then $K_{p_i} \neq \varnothing \Rightarrow i \in \{1, s\}$ [2].

5 K_p in finite simple groups

Theorem 2. *Let G is a finite simple group, $|G| = n, K_3 \neq \varnothing$. Then the involutions of G form one class of conjugate elements [3].*

The idea of proof
We relied on the following theorem 3 [4].

Theorem 3. *Let G is a simple finite group, $D \subset G$, $H = N_G(D), \varphi_1$ is the main character of H. If there are such non-main irreducible characters φ_i, φ_j of H, that $\Theta = \varphi_1 + \varphi_i - \varphi_j$ disappears on $H \setminus D$, then $\forall_G i\ \exists_H i_1\ (i \sim i_1)$.*

We have:

1. $k \in K_3, D = \{k, k^2\}$:
   - $\forall_G\ i_1, i_2\ (i_1 i_2 \in D \to i_2 i_1 \in D)$;
   - $\forall g \notin N_G(D)\ (D \cap D^g = \varnothing)$.
2. $H = N_G(D) \Rightarrow H \simeq S_3 \Rightarrow H \setminus D = \{e, i_1, i_2, i_3\}$.
3. The table of irreducible characters of S_3

S_3	(1 2)	(1 2 3)	e
χ_1	1	1	1
χ_2	-1	1	1
χ_3	0	-1	2

4. $\Theta(e) = (\chi_1 + \chi_2 - \chi_3)(e) = 1 + 1 - 2 = 0$;
   $\Theta(i) = (\chi_1 + \chi_2 - \chi_3)(i) = 1 - 1 - 0 = 0$.

According to the theorem 3, we obtain that the involutions of G form one class of conjugate elements.

6 K_5 in the projective special linear group $L_3(4)$

Using the properties of the set K_5, it is shown that the groups $L_3(4)$ and A_8 are not isomorphic, although they have the same order [2].

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