Preserving properties by types and expansions of structures

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Introduction

- We study various properties which are preserved under given conditions. These preservations generalize the notion of (p, q)-preserving formula and its variations for correspondent type-definable sets, with T.E. Rajabov.
- We introduce and study some general principles and hierarchical properties of expansions and restrictions of structures and their theories. These principles are based on upper and lower cones, lattices, and permutations. The general approach is applied to describe these properties for classes of ω-categorical theories and structures, Ehrenfeucht theories and their models, strongly minimal, ω_1-categorical, and stable ones.
- We study some hierarchy properties of expansions and restrictions of structures with given degrees of rigidity, with B.Sh. Kulpeshov.

Definition

Let $\mathcal{M}$ be a structure, $P_1 \subseteq M^{k_1}, \ldots, P_n \subseteq M^{k_n}$, $Q \subseteq M^m$ be properties, $\Phi = \Phi(\overline{x}_1, \ldots, \overline{x}_n, \overline{y})$ be a type with $l(\overline{x}_1) = k_1, \ldots$, $l(\overline{x}_n) = k_n$, $l(\overline{y}) = m$. We say that the tuple $(P_1, \ldots, P_n, Q)$ is (*totally*) Φ-*preserved*, or Φ is (*totally*) $(P_1, \ldots, P_n, Q)$-*preserving*, if for any $\overline{a}_1 \in P_1, \ldots, \overline{a}_n \in P_n$,

$$\Phi(\overline{a}_1, \ldots, \overline{a}_n, \mathcal{M}) \subseteq Q.$$

Here we also say on *universal* Φ- and $(P_1, \ldots, P_n, Q)$-preservation. If $\Phi(\overline{a}_1, \ldots, \overline{a}_n, \mathcal{M}) \subseteq Q$ for some $\overline{a}_1 \in P_1, \ldots, \overline{a}_n \in P_n$, then we say that $(P_1, \ldots, P_n, Q)$ is *existentially* Φ-*preserved*, or Φ is *existentially* $(P_1, \ldots, P_n, Q)$-*preserving*.

Definition

If $\Phi(\overline{a}_1, \ldots, \overline{a}_n, \mathcal{M}) \cap Q \neq \emptyset$ for some $\overline{a}_1 \in P_1, \ldots, \overline{a}_n \in P_n$ and $(P_1, \ldots, P_n, Q)$ is not existentially Φ-preserved by these tuples $\overline{a}_i$, then we say that $(P_1, \ldots, P_n, Q)$ is *$\exists$-partially Φ-preserved*, or Φ is *$\exists$-partially $(P_1, \ldots, P_n, Q)$-preserving*. If this property holds for any $\overline{a}_1 \in P_1, \ldots, \overline{a}_n \in P_n$, we say that $(P_1, \ldots, P_n, Q)$ is *$\forall$-partially Φ-preserved*, or Φ is *$\forall$-partially $(P_1, \ldots, P_n, Q)$-preserving*.

We say that the tuple $(P_1, \ldots, P_n, Q)$ is *$\exists$-partially Φ-non-preserved*, or Φ is *$\exists$-partially $(P_1, \ldots, P_n, Q)$-non-preserving*, if $\Phi(\overline{a}_1, \ldots, \overline{a}_n, \mathcal{M}) \cap \overline{Q} \neq \emptyset$ for some $\overline{a}_1 \in P_1, \ldots, \overline{a}_n \in P_n$, where $\overline{Q} = M^m \setminus Q$. If this property holds for any $\overline{a}_1 \in P_1, \ldots, \overline{a}_n \in P_n$, we say that $(P_1, \ldots, P_n, Q)$ is *$\forall$-partially Φ-non-preserved*, or Φ is *$\forall$-partially $(P_1, \ldots, P_n, Q)$-non-preserving*.

Definition

We say that the tuple $(P_1, \ldots, P_n, Q)$ is *totally Φ-non-preserved*, or Φ is *totally $(P_1, \ldots, P_n, Q)$-non-preserving*, if $\Phi(\overline{a}_1, \ldots, \overline{a}_n, \mathcal{M}) \cap \overline{Q} \neq \emptyset$ for any $\overline{a}_1 \in P_1, \ldots, \overline{a}_n \in P_n$.

If $\Phi(\overline{a}_1, \ldots, \overline{a}_n, \mathcal{M}) \subseteq \overline{Q}$ for some $\overline{a}_1 \in P_1, \ldots, \overline{a}_n \in P_n$, then we say that $(P_1, \ldots, P_n, Q)$ is *existentially Φ-disjoint*, or Φ is *existentially $(P_1, \ldots, P_n, Q)$-disjointing*. If $\Phi(\overline{a}_1, \ldots, \overline{a}_n, \mathcal{M}) \subseteq \overline{Q}$

for any $\overline{a}_1 \in P_1, \ldots, \overline{a}_n \in P_n$, then we say that $(P_1, \ldots, P_n, Q)$ is *totally Φ-disjoint* or *universally Φ-disjoint*, or Φ is *totally $(P_1, \ldots, P_n, Q)$-disjointing*, or *universally $(P_1, \ldots, P_n, Q)$-disjointing*.

Definition

If Φ is a singleton $\{\varphi\}$ then totally/existentially/partially Φ-(non-)preserved/disjoint tuples are called *totally/existentially/partially* φ-*(non-)preserved/disjoint*, respectively, and φ is *totally/existentially/partially* $(P_1, \ldots, P_n, Q)$-*(non-)preserving/disjointing*.

If $P_1 = \ldots = P_n = Q$ then $(P_1, \ldots, P_n, Q)$-preserving type Φ is called $(P_1, \ldots, P_n, Q)$-*idempotent* and $(P_1, \ldots, P_n, Q)$ is Φ-*idempotent*. If $\Phi = \{\varphi\}$ then we replace Φ by φ in the definition of idempotency.

Proposition (Rajabov – S.)

1. If a type Φ is totally $(P_1, \ldots, P_n, Q)$-preserving/disjointing and $P_1 \times \ldots \times P_n \neq \emptyset$ then Φ is existentially $(P_1, \ldots, P_n, Q)$-preserving/disjointing.
2. If a type Φ is $\forall$-partially $(P_1, \ldots, P_n, Q)$-(non-)preserving and $P_1 \times \ldots \times P_n \neq \emptyset$ then Φ is $\exists$-partially $(P_1, \ldots, P_n, Q)$-(non-)preserving.

Proposition (Rajabov – S.)

For any type Φ and definable or non-definable relations $P_1, \ldots, P_n, Q$ in a structure $\mathcal{M}$ the following conditions are equivalent:
1) Φ is totally/existentially $(P_1, \ldots, P_n, Q)$-preserving;
2) Φ is totally/existentially $(P_1, \ldots, P_n, \overline{Q})$-disjointing.

Proposition (Rajabov – S.)

For any $(P_1, \ldots, P_n, Q)$ and Φ, $(P_1, \ldots, P_n, Q)$ is totally/existentially/partially Φ-(non-)preserved/disjoint iff $(P_1 \times \ldots \times P_n, Q)$ is totally/existentially/partially Φ-(non-)preserved/disjoint, where $(\overline{x}_1, \ldots, \overline{x}_n, \overline{y})$ in Φ is replaced by $(\overline{x}_1\hat{\ }\ldots\hat{\ }\overline{x}_n, \overline{y})$.

Definition. Let T be a complete theory, $\mathcal{M} \models T$. Consider types $p(\overline{x}), q(\overline{x}) \in S(\emptyset)$, realized in $\mathcal{M}$, and all (p, q)-*preserving* formulae $\varphi(\overline{x}, \overline{y})$ of T, i. e., formulae for which there is $\overline{a} \in M$ such that $\models p(\overline{a})$ and $\varphi(\overline{a}, \overline{y}) \vdash q(\overline{y})$. For each such a formula $\varphi(\overline{x}, \overline{y})$, we define a relation $R_{p,\varphi,q} \rightleftharpoons \{(\overline{a}, \overline{b}) \mid \mathcal{M} \models p(\overline{a}) \wedge \varphi(\overline{a}, \overline{b})\}$. If $(\overline{a}, \overline{b}) \in R_{p,\varphi,q}$, then the pair $(\overline{a}, \overline{b})$ is called a (p, φ, q)-*arc*.

Proposition (Rajabov – S.)

For any types $p(\overline{x}), q(\overline{y}) \in S(\emptyset)$ and a formula $\varphi(\overline{x}, \overline{y})$ the following conditions are equivalent:
1) the formula φ is (p, q)-preserving;
2) the pair $(p(\mathcal{M}),\ q(\mathcal{M}))$ is totally φ-preserved;
3) the pair $(p(\mathcal{M}),\ q(\mathcal{M}))$ is existentially φ-preserved.

Proposition (Rajabov – S.)

If some conjunction of formulae in a type Φ is totally/existentially $(P_1, \ldots, P_n, Q)$-preserving/disjoint then Φ is totally/existentially $(P_1, \ldots, P_n, Q)$-preserving/disjoint.

Proposition (Rajabov – S.)

If a type Φ is α-partially $(P_1, \ldots, P_n, Q)$-(non-)preserving, where $\alpha \in \{\forall, \exists\}$, then any conjunction of formulae in a type Φ is α-partially $(P_1, \ldots, P_n, Q)$-(non-)preserving.

Proposition (Rajabov – S.)

If properties $P_1, \ldots, P_n, Q$ are type-definable in a saturated structure then a type Φ is partially $(P_1, \ldots, P_n, Q)$-preserving /disjoint iff some conjunction of formulae in Φ is totally/existentially/partially $(P_1, \ldots, P_n, Q)$-preserving/disjoint.

Proposition (Monotony)

If $(P_1, \ldots, P_n, Q)$ is Φ-preserved, $P_1 \supseteq P_1', \ldots, P_n \supseteq P_n'$, $Q \subseteq Q'$, $\Phi \subseteq \Phi'$ then $(P_1', \ldots, P_n', Q')$ is Φ'-preserved.

Notation

For types Φ and Ψ we denote by $\Phi \vee \Psi$ the type $\{\varphi \vee \psi \mid \varphi \in \Phi, \psi \in \Psi\}$, and by $\Phi \wedge \Psi$ the type $\Phi \cup \Psi$, if it is consistent.

Proposition (Union)

If $(P_1, \ldots, P_n, Q)$ is Φ-preserved and $(P_1, \ldots, P_n, Q')$ is Ψ-preserved, with $Q, Q' \subseteq M^m$, then $\Phi \vee \Psi$ is $(P_1, \ldots, P_n, Q \cup Q')$-preserving and $\Phi \wedge \Psi$, if it is consistent, is $(P_1, \ldots, P_n, Q \cap Q')$-preserving.

Corollary (Rajabov – S.)

If there is a $(P_1, \ldots, P_n, Q)$-preserving type Φ then the set $Z_\Phi(P_1, \ldots, P_n, Q)$ of all $(P_1, \ldots, P_n, Q)$-preserving types, which are contained in Φ, forms a distributive lattice $\langle Z_\Phi(P_1, \ldots, P_n, Q); \vee, \wedge\rangle$ with the least element Φ.

Traces of types

Definition. Let $\Phi = \Phi(\overline{x}_1, \dots, \overline{x}_n, \overline{y})$ be a type with consistent $\Phi(\overline{a}_1, \dots, \overline{a}_n, \overline{y})$, where $\overline{a}_1, \dots, \overline{a}_n$ be tuples in a model $\mathcal{M}$ of a given theory T. A *trace* of Φ with respect to $(\overline{a}_1, \dots, \overline{a}_n)$, or a Φ-*trace*, is a family $\{Q_i \subseteq M^{l(\overline{y})} \mid i \in I\}$ such that $\Phi(\overline{a}_1, \dots, \overline{a}_n, \mathcal{M}) \subseteq \bigcup\limits_{i \in I} Q_i$ and Φ is ($\forall$-) $\exists$-partially $(\{\overline{a}_1\}, \dots, \{\overline{a}_n\}, Q_i)$-preserving for each $i \in I$.
If the sets Q_i are pairwise disjoint then the Φ-trace $\{Q_i \mid i \in I\}$ is *disjoint*, too.
The Φ-trace $\{Q_i \mid i \in I\}$ is called A-(*type*-)*definable* if each Q_i is a (type-)definable set, which are defined over A. We say on the (type-)definability of the trace is it is A-(*type*-)*definable* for some A.

Traces of types

Any type $\Phi(\overline{x}_1, \ldots, \overline{x}_n, \overline{y})$ has a type-definable trace $\{\Phi(\overline{a}_1, \ldots, \overline{a}_n, \mathcal{M})\}$ over the set $\cup\overline{a}_1 \cup \ldots \cup \cup\overline{a}_n$, which is a singleton. Similarly each type $\Theta(\overline{y})$ with $\Phi(\overline{a}_1, \ldots, \overline{a}_n, \mathcal{M}) \vdash \Theta(\overline{y})$ produce a singleton-trace $\{\Theta(\mathcal{M})\}$ for Φ. By the definition all these traces are disjoint.

Proposition

For any type $\Psi = \Phi(\bar{a}_1, \ldots, \bar{a}_n, \bar{y})$ the family $[\Psi]$ is a disjoint $\emptyset$-type-definable Φ-trace. It is definable iff each type $p(\bar{y})$ for $[\Psi]$ is isolated.

Corollary

If the model $\mathcal{M}$ is atomic then each Φ-trace $[\Psi]$ is definable.

Corollary

If the model $\mathcal{M}$ is saturated then each Φ-trace $[\Psi]$ is definable iff $\mathrm{Th}(\mathcal{M})$ is ω-categorical.

Theorem

For any types $\Phi_i = \Phi_i(\overline{x}_1, \ldots, \overline{x}_n, \overline{y}_i)$, $i = 1, \ldots, m$, $\Psi = \Psi(\overline{y}_1, \ldots, \overline{y}_n, \overline{z})$ and tuples $\overline{a}_1, \ldots, \overline{a}_n$ for $\overline{x}_1, \ldots, \overline{x}_n$, respectively, the trace $[S(\Phi_1, \ldots, \Phi_m, \Psi)(\overline{a}_1, \ldots, \overline{a}_n, \overline{z})]$, for a saturated structure $\mathcal{M}$, consists of all types $p(\overline{z}) \in S^{l(\overline{z})}(\emptyset)$ consistent with $\Psi(\overline{b}_1, \ldots, \overline{b}_m, \overline{z})$, where $\mathrm{tp}(\overline{b}_i) \in [\Phi_i(\overline{a}_1, \ldots, \overline{a}_n, \overline{y}_i)]$, $i = 1, \ldots, m$.

Preservations of properties by special formulae and types

Proposition (Rajabov – S.)

If φ is an atomic formula $f(x_1,\dots,x_n)\approx y$ then a tuple $(P_1,\dots,P_n,Q)=(P,\dots,P,P)$ with $\emptyset\neq P\subseteq M$ is φ-preserved, i.e. φ-idempotent, iff P is the universe of a subalgebra of a restriction of $\mathcal{M}$ till the signature symbol f.

Corollary (Rajabov – S.)

If Φ is the family of all atomic formula $f(x_1,\dots,x_n)\approx y$ for any functional signature symbol f of a structure $\mathcal{M}$ then a tuple $(P_1,\dots,P_n,Q)=(P,\dots,P,P)$ with $\emptyset\neq P\subseteq M$ is Φ-preserved, i.e. Φ-idempotent, iff P is the universe of a substructure of $\mathcal{M}$.

Proposition (Rajabov – S.)

If $\varphi(x_1^1, x_1^2; \ldots; x_n^1, x_n^2; y^1, y^2)$ is a formula

$$E(y^1, y^2) \wedge f(x_1^1, \ldots, x_n^1) \approx y^1 \wedge f(x_1^2, \ldots, x_n^2) \approx y^2$$

then a tuple $(P_1, \ldots, P_n, Q) = (E, \ldots, E, E)$ with an equivalence relation $E \subseteq M^2$ is φ-preserved, i.e. φ-idempotent, iff E is a congruence relation of a restriction of $\mathcal{M}$ till the signature symbol f.

Constructions of models of theories on a base of preservations of properties

Theorem

For any expansion $\mathcal{M}$ of a model $\mathcal{M}_0$ of a theory T_0 by naming each elements by infinitely many constants, with $T = \mathrm{Th}(\mathcal{M})$, the following conditions are equivalent:
(1) $\mathcal{M}$ satisfies the preserving condition: : a $\Sigma(T_0)$-formula $\varphi(x_1, \ldots, x_n, x)$ is $(\forall$-$)$ $\exists$-partially $(\{[c_1]\}, \ldots, \{[c_n]\}, M)$-preserving whenever $\exists x \varphi(c_1, \ldots, c_n, x) \in T$;
(2) $\mathcal{M} \restriction \Sigma(T_0)$ satisfies the preserving condition;
(3) $\mathcal{M}$ is a canonical model of a completion of T_0.

Preservations of properties and Tarski-Vaught test

Theorem

Let $\mathcal{N}$ be a substructure of a structure $\mathcal{M}$ in a signature Σ. Then the following conditions are equivalent:
(1) $\mathcal{N}$ is an elementary substructure of $\mathcal{M}$;
(2) for any formula $\varphi(x_1, \ldots, x_n, y)$ of the signature Σ and any elements $a_1, \ldots, a_n \in N$ if $\mathcal{M} \models \exists y\, \varphi(a_1, \ldots, a_n, y)$ then $\varphi(x_1, \ldots, x_n, y)$ is $(\forall$-$)$ $\exists$-partially $(\{a_1\}, \ldots, \{a_n\}, N)$-preserving;
(3) for any formula $\varphi(x_1, \ldots, x_n, y)$ of the signature Σ and any elements $a_1, \ldots, a_n \in N$ either $\varphi(x_1, \ldots, x_n, y)$ is $(\forall$-$)$ $\exists$-partially $(\{a_1\}, \ldots, \{a_n\}, M)$-disjoint or $\varphi(x_1, \ldots, x_n, y)$ is $(\forall$-$)$ $\exists$-partially $(\{a_1\}, \ldots, \{a_n\}, N)$-preserving;
(4) for any finite type $\Phi(x_1, \ldots, x_n, y)$ of the signature Σ and any elements $a_1, \ldots, a_n \in N$ either $\Phi(x_1, \ldots, x_n, y)$ is $(\forall$-$)$ $\exists$-partially $(\{a_1\}, \ldots, \{a_n\}, M)$-disjoint or $\Phi(x_1, \ldots, x_n, y)$ is $(\forall$-$)$ $\exists$-partially $(\{a_1\}, \ldots, \{a_n\}, N)$-preserving.

Definition. For a cardinality $\varkappa$, a *$\varkappa$-partite* graph is a graph whose vertices are (or can be) partitioned into $\varkappa$ disjoint independent sets, i.e. sets without arcs connecting elements inside these sets. Equivalently, it is a graph that can be colored with $\varkappa$ colors, so that no two endpoints of an arc have the same color. When $\varkappa = 2$ these are the *bipartite* graphs, when $\varkappa = 3$ they are called the *tripartite* graphs, etc.

Proposition

Let $\Gamma = \langle M; R\rangle$ be a graph. The following conditions are equivalent:
(1) Γ is $\varkappa$-partite;
(2) M is divided into disjoint parts P_i, $i < \varkappa$, such that the formula $R(x, y)$ is $(P_i, \overline{P_i})$-preserving for any $i < \varkappa$.

Corollary

Let $\Gamma = \langle M; R\rangle$ be a graph. The following conditions are equivalent:
(1) Γ is bipartite;
(2) there is $P \subseteq M$ such that the formula $R(x, y)$ is $(P, \overline{P})$-preserving and $(\overline{P}, P)$-preserving.

Corollary

Let $\Gamma = \langle M; R\rangle$ be a graph. The following conditions are equivalent:
(1) Γ is tripartite;
(2) there are disjoint $P_0, P_1, P_2 \subseteq M$ such that $M = P_0 \cup P_1 \cup P_2$ and the formula $R(x, y)$ is $(P_i, \overline{P_i})$-preserving for each $i < 3$.

Proposition

Let $\Gamma = \langle M; R\rangle$ be a graph. The following conditions are equivalent:
(1) $R = \emptyset$ (respectively, $R = M^2$);
(2) the formula $R(x, y)$ ($\neg R(x, y)$) is $(M, \emptyset)$-preserving;
(3) the formula $R(x, y)$ ($\neg R(x, y)$) is (M, M)-disjoint.

Using the assertions above one can describe series of type-definable structures, in particular, classes of (ordered) semigroups, groups, rings and fields, including spherically ordered ones, rectangular bands of groups, graded algebras, etc., their subalgebras and quotients.

Definition

Below we consider *regular* structures, i.e. relational structures without repetitions of interpretations of signature symbols. Let $\mathcal{M}$ be a regular structure, $\overline{\mathcal{M}}$ be a maximal regular expansion of $\mathcal{M}$ preserving the universe M. We denote by $B(\mathcal{M})$ the set of all restrictions of $\overline{\mathcal{M}}$ preserving the universe M. The set-theoretic operations on the Boolean $\mathcal{P}(\Sigma(\overline{\mathcal{M}}))$, forming its Cantor algebra, induce the *regular* atomic Boolean algebra $\mathcal{B}(\mathcal{M})$ on $B(\mathcal{M})$, with the greatest element $\overline{\mathcal{M}}$, the least element $\mathcal{M}_0$ with the empty signature, and $|\Sigma(\overline{\mathcal{M}})|$ atoms each of which has exactly one signature symbol. Here unions $\mathcal{N}_1 \cup \mathcal{N}_2$ and intersections $\mathcal{N}_1 \cap \mathcal{N}_2$, for $\mathcal{N}_1, \mathcal{N}_2 \in B(\mathcal{M})$, preserve the universe M and consists of unions of their signature relations, common signature symbols, respectively. The unions can be considered both as *combinations* and *fusions*, in a broad sense, of structures.

Definition

Recall that for a lattice $\mathcal{L}$ and its element a the *upper cone*, denoted by $\triangledown(a)$ and $\triangledown_a$, consists of all elements b in L with $a \leq b$, and the *lower cone*, denoted by $\triangle(a)$ and $\triangle_a$, consists of all elements b in L with $b \leq a$.

Theorem

The family $P_{\omega\text{-cat}} \subseteq B(\mathcal{M})$ of countably categorical structures is represented as the union of lower cones of all their elements and all these elements are not maximal. This family is closed under permutations and not closed under unions.

Theorem

Any Boolean algebra $\mathcal{B}(\mathcal{M})$ with a countable universe M contains structures with the property $P_{\rm Ehr}$ of Ehrenfeuchtness (i.e. the property of all structures whose elementary theories have finitely many and at least three countable models), without the least and the greatest elements of $\mathcal{B}(\mathcal{M})$. This property is closed under permutations and can fail under restrictions and expansions. There are infinite chains alternating the Ehrenfeuchtness and the complement of this property. There are atomic structures $\mathcal{N} \in B(\mathcal{M})$ belonging to $P_{\rm Ehr}$.

Recall that a structure $\mathcal{N}$ is called *strongly minimal* if for any $\mathcal{N}' \equiv \mathcal{N}$ and any formula $\varphi(x, \overline{a})$ in the language of $\mathcal{N}$ with parameters $\overline{a} \in N'$ the set $\varphi(\mathcal{N}', \overline{a}) = \{b \mid \mathcal{N}' \models \varphi(b, \overline{a})\}$ is either finite or cofinite in N. A theory T is called *strongly minimal* if $T = \mathrm{Th}(\mathcal{N})$ for a strongly minimal structure $\mathcal{N}$.

Theorem

Any Boolean algebra $\mathcal{B}(\mathcal{M})$ with an infinite universe M contains a distributive sublattice $\mathcal{B}_{\rm sm}(\mathcal{M})$ of all strongly minimal structures $\mathcal{N} \in B(\mathcal{M})$. This sublattice closed under permutations and forms a Boolean algebra with the least element $\mathcal{N}_0$ and the greatest element $\mathcal{SM}$ forming $\triangle(\mathcal{SM})$ which is equal to $B_{\rm sm}(\mathcal{M})$.

Theorem

Any Boolean algebra $\mathcal{B}(\mathcal{M})$ with an infinite universe M contains structures with the property $P_{\omega_1\text{-cat}}$ of ω_1-categoricity, including the least element of $\mathcal{B}(\mathcal{M})$. This property is closed under permutations and can fail under restrictions and expansions. There are infinite chains alternating the ω_1-categoricity and the complement of this property. There are structures $\mathcal{N} \in B(\mathcal{M})$ of finite signatures with $\nabla_{\mathcal{N}} \cap P_{\omega_1\text{-cat}} = \emptyset$.

Definition

Recall that a formula $\varphi(\overline{x}, \overline{y})$ of a theory T is called *stable* if there are no tuples $\overline{a}_i, \overline{b_i} \in N$, where $i \in \omega$, $\mathcal{N} \models T$, such that $\mathcal{N} \models \varphi(\overline{a}_i, \overline{b}_j) \Leftrightarrow i \leq j$. The theory T is called *stable* if all its formulae are stable. Models of a stable theory are called *stable*, too. Is a formula/theory/structure is not stable, it is called *unstable* or having the *order property*.

It is said that a formula $\varphi(\overline{x}, \overline{y})$ has the *strict order property* if there are parameters $\overline{a}_i \in N$, $i \in \omega$, such that the sets $\varphi(\overline{a}_i, \mathcal{N})$, $i \in \omega$, form a strictly descending chain with $\varphi(\overline{a}_i, \mathcal{N}) \supsetneqq \varphi(\overline{a}_{i+1}, \mathcal{N})$, $i \in \omega$.

It is said that an unstable formula $\varphi(\overline{x}, \overline{y})$ has the *independence property* if in every/some model $\mathcal{N}$ of T there is, for each $n \in \omega$, a family of tuples $\overline{a}_i$, $i \in n$, such that for each of the 2^n subsets X of n there is a tuple $\overline{b} \in N$ for which $\mathcal{N} \models \varphi(\overline{a}_i, \overline{b}) \Leftrightarrow i \in X$.

Theorem

The family $P_{\rm st} \subseteq B(\mathcal{M})$ of stable structures is represented as the union of lower cones of all its elements. This family is closed under permutations and not closed under unions, and these unions can produce both the strict order property and the independence property.

Definition

For a set A in a structure $\mathcal{N}$, $\mathcal{N}$ is called *semantically A-rigid* or *automorphically A-rigid* if any A-automorphism $f \in \mathrm{Aut}(\mathcal{N})$ is identical. The structure $\mathcal{N}$ is called *syntactically A-rigid* if $N = \mathrm{dcl}(A)$.

A structure $\mathcal{N}$ is called $\forall$*-semantically* / $\forall$*-syntactically n-rigid* (respectively, $\exists$*-semantically* / $\exists$*-syntactically n-rigid*), for $n \in \omega$, if $\mathcal{N}$ is semantically / syntactically A-rigid for any (some) $A \subseteq N$ with $|A| = n$.

Definition

The least n such that $\mathcal{N}$ is Q-semantically / Q-syntactically n-rigid, where $Q \in \{\forall, \exists\}$, is called the *$Q$-semantical / Q-syntactical degree of rigidity*, it is denoted by $\deg_{\rm rig}^{Q\text{-sem}}(\mathcal{N})$ and $\deg_{\rm rig}^{Q\text{-synt}}(\mathcal{N})$, respectively. Here if a set A produces the value of Q-semantical / Q-syntactical degree then we say that A *witnesses* that degree. If such n does not exists we put $\deg_{\rm rig}^{Q\text{-sem}}(\mathcal{N}) = \infty$ and $\deg_{\rm rig}^{Q\text{-synt}}(\mathcal{N}) = \infty$, respectively.

$$\deg_4(\mathcal{M}) \rightleftharpoons \left(\deg_{\rm rig}^{\exists\text{-sem}}(\mathcal{M}), \deg_{\rm rig}^{\exists\text{-synt}}(\mathcal{M}), \deg_{\rm rig}^{\forall\text{-sem}}(\mathcal{M}), \deg_{\rm rig}^{\forall\text{-synt}}(\mathcal{M})\right)$$

Proposition

For any structures $\mathcal{N}_1, \mathcal{N}_2 \in B(\mathcal{M})$ and $A \subseteq M$ if $\mathcal{N}_1 \in \nabla(\mathcal{N}_2)$, i.e. $\mathcal{N}_1$ is an expansion of $\mathcal{N}_2$, then $\mathrm{Aut}(\langle \mathcal{N}_1, c_a \rangle_{a \in A}) \leqslant \mathrm{Aut}(\langle \mathcal{N}_2, c_a \rangle_{a \in A})$ and $\mathrm{dcl}_{\mathcal{N}_1}(A) \supseteq \mathrm{dcl}_{\mathcal{N}_2}(A)$.

Corollary

For any structures $\mathcal{N}_1, \mathcal{N}_2 \in B(\mathcal{M})$, $A \subseteq M$, $Q \in \{\forall, \exists\}$ if $\mathcal{N}_1 \in \nabla(\mathcal{N}_2)$ then $\mathrm{deg}_{\mathrm{rig}}^{Q\text{-sem}}(\mathcal{N}_1) \leq \mathrm{deg}_{\mathrm{rig}}^{Q\text{-sem}}(\mathcal{N}_2)$ and $\mathrm{deg}_{\mathrm{rig}}^{Q\text{-synt}}(\mathcal{N}_1) \leq \mathrm{deg}_{\mathrm{rig}}^{Q\text{-synt}}(\mathcal{N}_2)$. In particular, if $\mathcal{N}_2$ is semantically / syntactically A-rigid then $\mathcal{N}_1$ is semantically / syntactically A-rigid, too.

Let $P_{\rm semr}\subseteq B(\mathcal{M})$, $P_{\rm syntr}\subseteq B(\mathcal{M})$ be the properties of semantic / syntactic $\emptyset$-rigidity, respectively.

Proposition

For any structure $\mathcal{M}$, $P_{\rm semr}=P_{\rm syntr}$ iff M is finite.

Theorem (Kulpeshov – S.)

The family $P_{\rm semr} \subseteq B(\mathcal{M})$ of semantically rigid structures, respectively, the family $P_{\rm synt} \subseteq B(\mathcal{M})$ of syntactically rigid structures, is represented as the union of upper cones of all its elements. Each of these families is closed under permutations, contains some atomic elements of $\mathcal{B}(\mathcal{M})$, and closed under intersections iff $|M| = 1$.

Below we consider concatenations $\mathcal{M}_1 + \mathcal{M}_2$ of linearly ordered sets $\mathcal{M}_1$ and $\mathcal{M}_2$. Since in the Boolean algebra $\mathcal{B}(\mathcal{M})$ all structures have the same universe M, when considering $\mathcal{M}_1 + \mathcal{M}_2$ we assume that the original structures $\mathcal{M}_i$ on the set M have partial orders in which one connected component gave a linear order for $\mathcal{M}_i$, and all other components are singletons and there $|M_{3-i}|$ many of them, $i = 1, 2$. After connecting $\mathcal{M}_1$ and $\mathcal{M}_2$, considered as a union, two connected components are formed with respect to the relation $\leq_1 \cup \leq_2$ of orders $\leq_1$ and $\leq_2$ in $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. This relation is then replaced by its extension to the desired linear order for $\mathcal{M}_1 + \mathcal{M}_2$.

Proposition (Kulpeshov – S.)

Let $\mathcal{M}_1 = \langle M_1, < \rangle$ be a linear ordering with $\deg_4(\mathcal{M}_1) = (\infty, \infty, \infty, \infty)$. Then $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = (\infty, \infty, \infty, \infty)$ for any linear ordering $\mathcal{M}_2 = \langle M_2, < \rangle$.

Proposition (Kulpeshov – S.)

For any infinite linear orderings $\mathcal{M}_1 = \langle M_1, < \rangle$ and $\mathcal{M}_2 = \langle M_2, < \rangle$ with $\deg_4(\mathcal{M}_1) = \deg_4(\mathcal{M}_2) = (0, 0, 0, 0)$ the following holds: $\deg_4(\mathcal{M}_1 + \mathcal{M}_2)$ equals $(0, 0, 0, 0)$, $(1, 1, m, m)$ for some natural $m \geq 1$ or $(1, 1, \infty, \infty)$.

Proposition (Kulpeshov – S.)

For any natural $m_1, m_2 \geq 1$ and for any infinite linear orderings $\mathcal{M}_1 = \langle M_1, < \rangle$, $\mathcal{M}_2 = \langle M_2, < \rangle$ with $\deg_4(\mathcal{M}_1) = (m_1, m_1, \infty, \infty)$ and $\deg_4(\mathcal{M}_2) = (m_2, m_2, \infty, \infty)$ the following holds: $\deg_4(\mathcal{M}_1 + \mathcal{M}_2) = (m, m, \infty, \infty)$, where $m_1 + m_2 \leq m \leq m_1 + m_2 + 1$.