Finite CSA groups and generalizations

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The class of CSA groups is quite wide, for example

1. every fully residually free group,
2. every ultra-power of a CSA,
3. free product of two CSA groups without involutions.

They have very serious roles in the study of residually free groups, universal theory of non-abelian free groups, limit groups, exponential groups and equational domains in algebraic geometry over groups.

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Despite this simple definition, the class of CT groups has also a crucial role in the study of residually free groups and so it has a close connection with CSA groups.
Every CSA group is CT but the converse is not true. In the presence of residual freeness, both properties are equivalent, a theorem which has been proved by B. Baumslag:

Residually free groups. Proc. London Math. Soc. **17**(3), 1967.

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2. If G is CT, then Fine et al proved that G is not CSA iff G contains a non-abelian subgroup G_0 which contains a nontrivial abelian subgroup H that is normal in G_0.
3. For K a field, the group $PSL_2(K)$ is never CSA but is CT if $char(K) = 2$ or $char(K) = 0$ and -1 is not the sum of two squares.

Benjamin Fine et al: *On* CT *and* CSA *groups and related ideas*. J. Group Theory, 2016, **19**.

More properties

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1. CSA and CT are universal first order properties, and hence these classes are inductive and subgroup-closed.
2. If a non-abelian group G is residually free, then the following are equivalent:
   - G is fully residually free.
   - G is CT.
   - G is CSA.
   - G is universally free.

Generalization

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M. Shahryari: *On conjugate separability of nilpotent groups*. J. Group Theory, 2024, **23**.

Suppose $\mathfrak{X}$ is a variety of groups. A group G can be called $\mathfrak{X}\mathrm{T}$ then, iff for any two $\mathfrak{X}$-subgroups $K_1, K_2 \leq G$ the assumption $K_1 \cap K_2 \neq 1$ implies that $\langle K_1, K_2 \rangle$ is also an $\mathfrak{X}$-group.

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Similarly, we call a group G a $\mathrm{CS}\mathfrak{X}$ group if all of its maximal $\mathfrak{X}$-subgroups are malnormal.

Example: NT_k and CSN_k groups

We call a group NT_k (nilpotency transitive of class k) if for any two $\mathfrak{N}_k$-subgroups K_1 and K_2, the assumption $K_1 \cap K_2 \neq 1$ implies that $\langle K_1, K_2 \rangle$ is nilpotent of class at most k.

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Also a group G is CSN_k (conjugately separated nilpotent of class k) if and only if every maximal $\mathfrak{N}_k$-subgroup of G is malnormal.

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Also a group G is CSN_k (conjugately separated nilpotent of class k) if and only if every maximal $\mathfrak{N}_k$-subgroup of G is malnormal.
The case $k = 1$ obviously coincides with the ordinary CT and CSA groups. It is also easy to see that the property CSA implies CSN_k.

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6. Suppose A and B are CSN_k groups without elements of order 2. Then the free product $G = A * B$ is also CSN_k.
7. Every finite CSN_k group is nilpotent of class at most k.

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It is well-known that every finite CSA group is abelian. The known proofs usually employ representation theory of finite groups or classification of finite simple groups. We present an elementary proof for this fact, and then the main idea of this elementary proof will be applied for a wide class of $\mathrm{CS}\mathfrak{X}$ groups.

Finite *CSX* groups in general

Before presenting our elementary proof, we need to emphasis on the fact:

Theorem

Let $\mathfrak{X}$ be a class of groups which contains all cyclic groups. Then every finite $\mathrm{CS}\mathfrak{X}$ group belongs to $\mathfrak{X}$.

Proof. The proof of this general fact is not elementary.

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$$G = \bigcup_x A^x.$$

This shows that $A = G$, a contradiction.

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4- If G is CSA then $A \cap B = 1$ for all distinct maximal abelian subgroups A and B.
5- If A is a malnormal subgroup, then $N_G(A) = A$.

The proof ...

Now, suppose G is a finite CSA group but not abelian. Let A_1 be a maximal abelian subgroup, with distinct conjugates

$$A_1^{x_{1,1}}, A_1^{x_{1,2}}, A_1^{x_{1,3}}, \ldots$$

Then all of these subgroups are maximal abelian and so their intersection is 1.

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Then all of these subgroups are maximal abelian and so their intersection is 1. Suppose

$$G \neq \bigcup_j A_1^{x_{1,j}}.$$

Then there is another maximal abelian subgroup A_2, such that $A_2 \cap A_1^{x_{1,j}} = 1$, for all j.

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Hence the union

$$(\bigcup_j A_1^{x_{1,j}} \setminus 1) \cup (\bigcup_r A_2^{x_{2,r}} \setminus 1)$$

is disjoint.

The proof ...

Continue this way! At the end there will be a number n, maximal abelian subgroups

$$A_1, A_2, \dots, A_n,$$

and elements xij $(1 \leq i \leq n, 1 \leq j \leq k_i)$, such that the union

$$G \setminus 1 = \bigcup_{i=1}^{n} \bigcup_{j=1}^{k_i} (A_i^{x_{ij}} \setminus 1)$$

is disjoint.

The proof ...

Hence, we have

$$\begin{aligned} |G| &= \sum_{i=1}^{n}[G : N_G(A_i)](|A_i| - 1) + 1 \\ &= \sum_{i=1}^{n}[G : A_i](|A_i| - 1) + 1 \\ &= n|G| - \sum_{i=1}^{n}[G : A_i] + 1. \end{aligned}$$

Therefore, we have

$$\sum_{i=1}^{n}[G : A_i] = (n-1)|G| + 1.$$

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Note that if $i \neq j$, then a_i is not conjugate to a_j (otherwise $A_i \cap A_j \neq 1$).

Now, using the class equation, we have

$$|G| = |Z(G)| + \sum_{i=1}^{n} |\mathrm{Cl}_G(a_i)| + \cdots = 1 + \sum_{i=1}^{n} |\mathrm{Cl}_G(a_i)| + \cdots$$

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Therefore

$$\sum_{i=1}^{n} |\mathrm{Cl}_G(a_i)| \leq |G| - 1,$$

and hence

$$(n-1)|G| + 1 \leq |G| - 1$$

which implies that $n = 1$.

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So $G = \bigcup_j A_1^{x_{1,j}}$, and hence $G = A_1$. This completes the proof.
Note. With some modifications, we can use the same proof for more general cases. For example, we can show by the same argument:

Theorem

Every finite CSN_k group is nilpotent of class k.

More varieties admitting the same elementary proof

The same idea can be used to the following case:

Theorem

Let $\mathfrak{X}$ *be a variety which contains all abelian groups and suppose* $\mathrm{CS}\mathfrak{X} \subseteq \mathfrak{X}\mathrm{T}$. *Then every finite* $\mathrm{CS}\mathfrak{X}$ *group belongs to* $\mathfrak{X}$.

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Example. Let $\mathfrak{X} = \mathfrak{Y}_q \cap \mathfrak{B}_n$, such that

$$\mathfrak{Y}_q = \{G : \ G^q \subseteq Z(G)\}$$

and $\mathfrak{B}_n$ is the Burnside variety.

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and $\mathfrak{B}_n$ is the Burnside variety. Let $\gcd(n, q) = 1$. Then $\mathfrak{X}$ satisfies the requirements of the above theorem.

Thank you.