ON PREORDERS BETWEEN THE RUDIN-KEISLER AND COMFORT PREORDERS AND A MODEL-THEORETIC CHARACTERIZATION OF THE COMFORT PREORDER

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As usual, $\boldsymbol{\beta}X$ denotes the standard Čech–Stone compactification of the discrete space X, which we identify with the set of ultrafilters over X (see [2, 6]). We consider here ultrafilters over ω although most of our results remain true for ultrafilters over any infinite set X. The *Rudin–Keisler* preorder $\leq_{\mathrm{RK}}$ on $\boldsymbol{\beta}\omega$ is defined by letting $\mathfrak{u} \leq_{\mathrm{RK}} \mathfrak{v}$ iff there exists $f : \omega \to \omega$ such that $\widetilde{f}(\mathfrak{v}) = \mathfrak{u}$, where $\widetilde{f} : \boldsymbol{\beta}\omega \to \boldsymbol{\beta}\omega$ is the continuous extension of f. The *Comfort* preorder $\leq_{\mathrm{C}}$ on $\boldsymbol{\beta}\omega$ is defined by letting $\mathfrak{u} \leq_{\mathrm{C}} \mathfrak{v}$ iff any $\mathfrak{v}$-compact space is $\mathfrak{u}$-compact, where a space X is $\mathfrak{u}$-*compact* iff $\widetilde{f}(\mathfrak{u}) \in X$ for any $f : \omega \to X$. (See [2, 6] for more on ultrafilters and $\leq_{\mathrm{RK}}$, and [3, 4] for $\leq_{\mathrm{C}}$.)

For $\mathfrak{u}, \mathfrak{v} \in \boldsymbol{\beta}\omega$ and any ordinal α, define: $\mathfrak{u}\, R_0\, \mathfrak{v}$ iff $\mathfrak{u}$ is principal, $R_{<\alpha} = \bigcup_{\beta<\alpha} R_\beta$, and $\mathfrak{u}\, R_\alpha\, \mathfrak{v}$ iff there exists a continuous map $f : \boldsymbol{\beta}\omega \to \boldsymbol{\beta}\omega$ such that $f(\mathfrak{v}) = \mathfrak{u}$ and $f(n)\, R_{<\alpha}\, \mathfrak{v}$ for all $n < \omega$. The hierarchy is non-degenerate and lies between $\leq_{\mathrm{RK}}$ and $\leq_{\mathrm{C}}$ as stated in the following theorem.

Theorem 1. *$R_1 = \leq_{\mathrm{RK}}$; $R_{<\alpha} \subset R_\alpha$ for all $\alpha < \omega_1$; $R_{<\omega_1} = R_{\omega_1} = \leq_{\mathrm{C}}$.*

If X, Y are spaces and α is an ordinal, $f : X^\alpha \to Y$ is *right-continuous w.r.t.* $A \subseteq X$ iff for all $\beta < \alpha$ the shift $x \mapsto f(a_0, a_1, \ldots, x, b_{\beta+1}, b_{\beta+2}, \ldots)$ is continuous whenever $a_0, a_1, \ldots \in A$ and $b_{\beta+1}, b_{\beta+2}, \ldots \in X$. As shown in [8, 9], if $n < \omega$, X is discrete, and Y is compact Hausdorff, then every $f : X^n \to Y$ uniquely extends to $\widetilde{f} : (\boldsymbol{\beta}X)^n \to Y$ that is right-continuous w.r.t. X. This fact provides a canonical way to obtain, for an arbitrary first-order model $\mathfrak{A}$, its *ultrafilter extension* $\boldsymbol{\beta}\mathfrak{A}$ ([8, 9]), generalizing the well-known construction of ultrafilter extensions of semigroups comprehensively treated in [6]. (Some historical remarks can be found in [7].)

If $n < \omega$, the relations R_n can be redefined in terms of ultrafilter extensions of n-ary operations on ω as follows: $\mathfrak{u}\, R_n\, \mathfrak{v}$ iff there exists $f : \omega^n \to \omega$ such that $\widetilde{f}(\mathfrak{v}, \ldots, \mathfrak{v}) = \mathfrak{u}$. Moreover, $R_m \circ R_n = R_{nm}$ (so R_n are not preorders for $2 \leq n < \omega$). These observations can be expanded to all R_α by using ω-ary operations on ω. Such an operation is identified with a continuous map of the Baire space ω^ω into the discrete space ω; these maps admit a natural hierarchy ranked by countable ordinals.

Proposition 1. *Any continuous $f : \omega^\omega \to \omega$ uniquely extends to $\widetilde{f} : (\boldsymbol{\beta}\omega)^\omega \to \boldsymbol{\beta}\omega$ that is right-continuous w.r.t. ω (in other words, ω-ary operations on ω extend to such operations on $\boldsymbol{\beta}\omega$).*

Proposition 2. *Let $\alpha < \omega_1$ and $\mathfrak{u}, \mathfrak{v} \in \boldsymbol{\beta}\omega$. Then $\mathfrak{u}\, R_\alpha\, \mathfrak{v}$ iff there exists a continuous $f : \omega^\omega \to \omega$ of rank α such that $\widetilde{f}(\mathfrak{v}, \mathfrak{v}, \ldots) = \mathfrak{u}$.*

The composition of arbitrary $R_{<\alpha}$ is expressed via a multiplication-like operation on ordinals. To simplify notation, denote $\sup_{\gamma<\alpha}(\gamma\cdot\beta)$ by $({<}\alpha)\cdot\beta$; the explicit calculation of these ordinals, used in getting the following result, is rather cumbersome.

Theorem 2. *Let $\alpha, \beta < \omega_1$.*

(i) *$R_\alpha \circ R_\beta = R_\gamma$ where $\gamma = \beta \cdot \alpha$ if $\beta = 0$ or $\alpha < \omega$, $\gamma = \beta \cdot (\alpha + 1) - 1$ if $0 < \beta < \omega$ and $\alpha \geq \omega$, and $\gamma = \beta \cdot (\alpha + 1)$ if $\alpha, \beta \geq \omega$;*
(ii) *If $\alpha > 0$ is limit, then $R_{<\alpha} \circ R_\beta = R_{<\gamma}$ where $\gamma = \beta \cdot \alpha$;*
(iii) *If $\beta > 0$ is limit, then $R_\alpha \circ R_{<\beta} = R_{<\gamma}$ where $\gamma = ({<}\beta) \cdot \alpha$ if $\alpha < \omega$, and $\gamma = ({<}\beta) \cdot (\alpha + 1)$ otherwise;*
(iv) *If $\alpha, \beta > 0$ are limit, then $R_{<\alpha} \circ R_{<\beta} = R_{<\gamma}$ where $\gamma = ({<}\beta) \cdot \alpha$.*

Corollary 1. *Let $2 \leq \alpha \leq \omega_1$. Then $R_{<\alpha}$ is a preorder iff α is multiplicatively indecomposable.*

Define preorders between $\leq_{\mathrm{RK}}$ and $\leq_{\mathrm{C}}$ by letting $\leq_0 = \leq_{\mathrm{RK}}$ and $\leq_{1+\alpha} = R_{<\omega^{\omega^\alpha}}$ for all $\alpha \leq \omega_1$. So, if α is infinite, $R_{<\alpha} = \leq_\alpha$ iff α is an epsilon number. Also $\leq_\alpha \circ \leq_\beta = \leq_\gamma$ where $\gamma = \max(\alpha, \beta)$.

Let us now consider two applications in model theory. The first concerns ultrafilter extensions. As shown in [5], for any ultrafilter $\mathfrak{v}$ and semigroup S, the set $\{\mathfrak{u} : \mathfrak{u} \leq_{\mathrm{C}} \mathfrak{v}\}$ forms a subsemigroup of $\boldsymbol{\beta}S$. This can be expanded to arbitrary first-order models and relations $R_{<\alpha}$ as follows.

Corollary 2. *For all ordinals $\alpha > 1$, ultrafilters $\mathfrak{v}$, and models $\mathfrak{A}$, the following are equivalent:*

(i) $\{\mathfrak{u} : \mathfrak{u}\, R_{<\alpha}\, \mathfrak{v}\}$ *forms a submodel of the model* $\boldsymbol{\beta}\mathfrak{A}$;

(ii) α *is additively indecomposable.*

Consequently, for all $\alpha > 0$, $\mathfrak{v}$, and $\mathfrak{A}$, $\{\mathfrak{u} : \mathfrak{u} \le_\alpha \mathfrak{v}\}$ forms a submodel of $\boldsymbol{\beta}\mathfrak{A}$.

The second application concerns ultrapowers. For a model $\mathfrak{A}$ with the universe A, a set I, an ultrafilter $\mathfrak{u}$ over I, and $f : I \to A$, let $f_\mathfrak{u}$ denote the $\mathfrak{u}$-equivalence class of f, and $\prod_\mathfrak{u} \mathfrak{A}$ the ultrapower of $\mathfrak{A}$ by $\mathfrak{u}$. As shown in [1], the Rudin–Keisler preorder has a natural characterization in terms of ultrapowers: $\mathfrak{u} \le_{\mathrm{RK}} \mathfrak{v}$ iff $\prod_\mathfrak{u} \mathfrak{A} \preceq \prod_\mathfrak{v} \mathfrak{A}$ for all models $\mathfrak{A}$, and also iff $\prod_\mathfrak{u} \mathfrak{N} \preceq \prod_\mathfrak{v} \mathfrak{N}$ where $\mathfrak{N}$ is the *full model of* ω, i.e., it has the universe ω and all relations and operations on ω.

It is not difficult to characterize R_n with $n < \omega$ via ultrapowers in a similar manner:

Proposition 3. *For all $\mathfrak{u}, \mathfrak{v}$, and $n < \omega$, the following are equivalent:*

(i) $\mathfrak{u}\, R_n\, \mathfrak{v}$;

(ii) $\prod_\mathfrak{u} \mathfrak{A} \preceq \prod_\mathfrak{v} \ldots \prod_\mathfrak{v} \mathfrak{A}$ *(n times) for all models* $\mathfrak{A}$;

(iii) $\prod_\mathfrak{u} \mathfrak{N} \preceq \prod_\mathfrak{v} \ldots \prod_\mathfrak{v} \mathfrak{N}$ *(n times).*

To handle the case $\alpha \ge \omega$, the following "skew version" of limit ultrapowers is used. First for any $e : A \to B$ define $e^\mathfrak{u} : \prod_\mathfrak{u} A \to \prod_\mathfrak{u} B$ by letting $e^\mathfrak{u}(g_\mathfrak{u}) := (e \circ g)_\mathfrak{u}$. Clearly, $e : \mathfrak{A} \preceq \mathfrak{B}$ implies $e^\mathfrak{u} : \prod_\mathfrak{u} \mathfrak{A} \preceq \prod_\mathfrak{u} \mathfrak{B}$. Then for every model $\mathfrak{A}$, ultrafilter $\mathfrak{u}$, and ordinals α, define the models $\mathfrak{A}_{\mathfrak{u},\alpha}$ and their embeddings $e_{\beta\alpha}$ for $\beta < \alpha$:

(i) $\mathfrak{A}_{\mathfrak{u},0} := \mathfrak{A}$, $\mathfrak{A}_{\mathfrak{u},1} := \prod_\mathfrak{u} \mathfrak{A}$, and $e_{01} = d$ (the diagonal map);

(ii) if α is limit, then $\mathfrak{A}_{\mathfrak{u},\alpha} := \lim_{\beta\to\alpha} \mathfrak{A}_{\mathfrak{u},\beta}$ w.r.t. the system $\{e_{\gamma\beta}\}_{\gamma<\beta<\alpha}$, and the maps $e_{\beta\alpha}$ ($\beta < \alpha$) are defined naturally;

(iii) if $\alpha = \beta + 1$, then $\mathfrak{A}_{\mathfrak{u},\alpha} := \prod_\mathfrak{u} \mathfrak{A}_{\mathfrak{u},\beta}$, and the maps $e_{\delta\alpha}$ ($\delta < \alpha$) are defined as follows:

  (a) if $\beta = \gamma + 1$, then $e_{\beta\alpha} := e^\mathfrak{u}_{\gamma\beta}$;

  (b) if $\beta > 0$ is limit, then for any $g \in A_{\mathfrak{u},\beta}$
$$e_{\beta\alpha}(g) := e^\mathfrak{u}_{\gamma\beta}(h)$$
  for some $\gamma < \beta$ and $h \in g \cap A_{\mathfrak{u},\gamma+1}$;

  (c) for any $\delta < \beta$, $e_{\delta\alpha} := e_{\beta\alpha} \circ e_{\delta\beta}$.

Lemma 1. *All $\mathfrak{A}_{\mathfrak{u},\alpha}$ are well defined, and if $\beta < \alpha$ then $e_{\beta\alpha} : \mathfrak{A}_{\mathfrak{u},\beta} \preceq \mathfrak{A}_{\mathfrak{u},\alpha}$.*

Theorem 1. *For $\mathfrak{u}, \mathfrak{v} \in \boldsymbol{\beta}\omega$, and a limit ordinal $\alpha > 0$, the following are equivalent:*

(i) $\mathfrak{u}\, R_{<\alpha}\, \mathfrak{v}$;

(ii) *there is $\beta < \alpha$ such that $\mathfrak{A}_{\mathfrak{u},1} \preceq \mathfrak{A}_{\mathfrak{v},\beta}$ for all models $\mathfrak{A}$;*

(iii) *there is $\beta < \alpha$ such that $\mathfrak{N}_{\mathfrak{u},1} \preceq \mathfrak{N}_{\mathfrak{v},\beta}$.*

Consequently, $\mathfrak{u} \le_{\mathrm{C}} \mathfrak{v}$ iff $\mathfrak{A}_{\mathfrak{u},1} \preceq \mathfrak{A}_{\mathfrak{v},\omega_1}$ for all models $\mathfrak{A}$, and also iff $\mathfrak{N}_{\mathfrak{u},1} \preceq \mathfrak{N}_{\mathfrak{v},\omega_1}$.

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