NOTES ON THE CONGRUENCE DENSITIES AND QUASIORDER DENSITIES OF SUBLATTICES

Gábor Czédli*

University of Szeged, Bolyai Institute. Szeged, Aradi vértanúk tere 1, HUNGARY 6720

e-mail: czedli@math.u-szeged.hu, url: https://www.math.u-szeged.hu/~czedli/

Dedicated to Ferencné Piroska Varga, our esteemed librarian at the Mathematical Library of Bolyai Institute, on her sixty-fifth birthday, in honor of her excellent work and her academic background, holding M.Sc. degrees in library science and in mathematics and physics education.

Abstract. For a positive integer n, let $\mathrm{mnc}(n)$ denote the maximum number of congruences among all n-element lattices; that is, $\mathrm{mnc}(n) = \max\{|\mathrm{Con}(L)| : L \text{ is an } n\text{-element lattice}\}$, where $\mathrm{Con}(L)$ stands for the congruence lattice of L. We know from a 1997 paper of R. Freese that $\mathrm{mnc}(n) = 2^{n-1}$. The *congruence density* $\mathrm{cd}(L)$ of a finite lattice L is defined to be the quotient $|\mathrm{Con}(L)|/\mathrm{mnc}(|L|)$. That is, if an n-element lattice L has exactly k congruences, then $\mathrm{cd}(L) = k/2^{n-1}$. The maximum number of (compatible) quasiorders of an n-element lattice L is 2^{2n-2}, and we define the *quasiorder density* $\mathrm{qd}(L)$ of L—analogously to $\mathrm{cd}(L)$—as $\mathrm{qd}(L) := |\mathrm{Quo}(L)|/2^{2n-2}$, where $\mathrm{Quo}(L)$ is the quasiorder lattice of L. We prove that if S is a sublattice of a finite lattice L and at least one of the following three conditions holds: (i) L is modular; (ii) S is a cover-preserving sublattice of L; or (iii) L is a dismantlable extension of S, then $\mathrm{cd}(L) \le \mathrm{cd}(S)$ and $\mathrm{qd}(L) \le \mathrm{qd}(S)$.

1 Introduction

Every lattice occurring in this paper will be assumed to be finite, even when this assumption is not mentioned again. The paper assumes no more

*This research was supported by the National Research, Development and Innovation Fund of Hungary, under funding scheme K 138892. Version dated **May 13, 2025**

than a minimal familiarity with lattices or universal algebra. For a lattice $L = (L; \vee, \wedge)$, $\mathrm{Con}(L)$ will denote the congruence lattice of L. Accordingly, $|\mathrm{Con}(L)|$ stands for the number of congruence relations of L. By Freese [8], the largest possible value of $|\mathrm{Con}(L)|$ for n-element lattices L is $\mathrm{mnc}(n) = 2^{n-1}$. In other words, $\mathrm{mnc}(n) := \max\{|\mathrm{Con}(L)| : L \text{ is an } n\text{-element lattice}\}$ is equal to 2^{n-1}. The second, third, fourth, etc. largest values were determined by Czédli [2], [4], and Mureşan and Kulin [13]. Following Czédli [4], we define the *congruence density* $\mathrm{cd}(L)$ of a finite lattice L as the quotient $\mathrm{cd}(L) := |\mathrm{Con}(L)|/\mathrm{mnc}(|L|)$. Clearly, $0 < \mathrm{cd}(L) \leq 1$ and, by Freese [8], $\mathrm{cd}(L) = 1$ if and only if L is a chain. Therefore, in some vague sense, the congruence density measures how close a lattice is to being a chain. If a sublattice S of L is far from being a chain, then so is L itself. Some evidence supporting this idea is implicit in Czédli [4]; namely, whenever S is a sublattice of a finite lattice L, $8 \leq |S|$, and $1/8 + 3/2^{|S|-1} \leq \mathrm{cd}(S)$, then $\mathrm{cd}(L) \leq \mathrm{cd}(S)$. This fact and our experience with Czédli [3] and [4] lead to the problem: Does the inequality $\mathrm{cd}(L) \leq \mathrm{cd}(S)$ hold for every finite lattice L and every sublattice S of L? In Theorem 2.1, we provide a positive answer in three particular cases.

A *quasiorder* on a lattice L is a compatible preorder, that is, a compatible, reflexive, transitive relation on L. The *quasiorder lattice* $\mathrm{Quo}(L) = (\mathrm{Quo}(L); \subseteq)$ of L is the lattice of all quasiorders on L; note that $\mathrm{Con}(L)$ is a sublattice of $\mathrm{Quo}(L)$. Analogously to the case of congruences, we denote the *maximum number of quasiorders* on an n-element lattice by $\mathrm{mnq}(n) := \max\{|\mathrm{Quo}(L)| : L \text{ is an } n\text{-element lattice}\}$, and we define the *quasiorder density* of a finite lattice L as

$$\mathrm{qd}(L) := |\mathrm{Quo}(L)|/\mathrm{mnq}(|L|). \tag{1.1}$$

2 Stating the results

We say that a sublattice S of a finite lattice L is a *cover-preserving sublattice* of L if for every $x, y \in S$, whenever y covers x in S (denoted by $x \prec_S y$), then y covers x in L as well (denoted by $x \prec_L y$ or simply $x \prec y$). A *proper sublattice* of L is a sublattice that is distinct from L. For a sublattice K of a finite lattice L, we say that L is a *dismantlable extension* of K if there exists a sequence $T_{|K|}, T_{|K|+1}, \dots, T_{|L|}$ of sublattices of L such that $T_{|K|} = K$, $T_{|L|} = L$, and for every $i \in \{|K|+1, \dots, |L|\}$, T_{i-1} is a proper sublattice of T_i and $|T_i| = i$. Note that L is a *dismantlable lattice* in the well-known classical sense of Baker, Fishburn, and Roberts [1] if and only if L is a dismantlable extension of one of its one-element sublattices. The diagram on the left of Figure 1 exemplifies that a dismantlable lattice L need not be a dismantlable extension of each of its sublattices K. Our main goal is to prove the following theorem.

Theorem 2.1. *Let S be a sublattice of a finite lattice L, and assume that at least one of the following three conditions is satisfied:*

1. *S is a cover-preserving sublattice of L,*
2. *L is modular, or*
3. *L is a dismantlable extension of S.*

Then $\mathrm{cd}(L) \le \mathrm{cd}(S)$.

For a lattice L, Czédli and Szabó [5] proved[1] that $\mathrm{Quo}(L)$ is isomorphic to the direct square $\mathrm{Con}(L)^2$. Consequently, $\mathrm{mnq}(n) = \mathrm{mnc}(n)^2 = 2^{2n-2}$, and (1.1) defining the quasiorder density of a finite lattice L simplifies to $\mathrm{qd}(L) := |\mathrm{Quo}(L)|/2^{2|L|-2}$. Utilizing the isomorphism $\mathrm{Quo}(L) \cong \mathrm{Con}(L)^2$, one can immediately see that the results of Czédli [3], [4], and the present paper, along with those proved in Mureşan and Kulin [13], directly imply their "quasiorder-counterparts". For example, the following statement follows trivially from Theorem 2.1 and the isomorphism $\mathrm{Quo}(L) \cong \mathrm{Con}(L)^2$.

Corollary 2.2. *If S is a sublattice of a finite lattice L and at least one of the conditions* (1)*,* (2)*, or* (3) *in Theorem 2.1 holds, then* $\mathrm{qd}(L) \le \mathrm{qd}(S)$.

We devote the rest of the paper to the proof of Theorem 2.1.

3 Preparatory concepts, notations, and lemmas

For a relation $\rho \subseteq X^2$ and a subset Y of X, the *restriction* of ρ to Y will be denoted by $\rho\rceil_Y$. That is, $\rho\rceil_Y = \rho \cap Y^2$. However, we do not always explicitly indicate when a relation is restricted. For example, we usually write "$\le$" or opt for an alternative notation rather than "$\le\rceil_Y$". A pair $(a, b) \in L^2$ is a *covering pair* (of elements of a lattice L) if the interval $[a, b] = \{x \in L : a \le x \le b\}$ is 2-element. Let $\mathrm{Cp}(L)$ denote the covering pairs of L. To recall some other notation and terminology for elements $a, b \in L$, note that the following seven statements are equivalent: $a \prec b$, b *covers* a, b is a *cover* of a, a is a *lower cover* of b, $[a, b]$ is a *prime interval*,

[1] For some historical comments on [5], see Davey [6].

(a,b) is an *edge*, and $(a,b) \in \mathrm{Cp}(L)$. For $(a,b) \in L^2$, the least congruence collapsing a and b will be denoted by $\mathrm{con}(a,b)$. The *poset of join-irreducible elements* of L is denoted by $\mathrm{Ji}(L)$; an $x \in L$ is said to be join-irreducible if it has exactly one lower cover. For $x \in \mathrm{Ji}(L)$, the unique lower cover of x will be denoted by x^- or, if L needs to be specified, by $x^{L,-}$. If an $x \in L$ has at least two lower covers, then x is *join-reducible*; the *set of join-reducible elements* will be denoted by $\mathrm{Jr}(L)$. Since $0 = 0_L$ has no lower cover at all, L is the disjoint union of $\{0\}$, $\mathrm{Ji}(L)$, and $\mathrm{Jr}(L)$. The *poset* $\mathrm{Mi}(L)$ of *meet-irreducible elements*, the set $\mathrm{Mr}(L)$ of *meet-reducible* elements, and the unique cover $x^+ = x^{L,+}$ of an $x \in \mathrm{Mi}(L)$ are defined dually.

For pairs $(a,b),(c,d) \in L^2$, we say that (a,b) is *prime-perspective up* to (c,d), in notation $(a,b) \overset{\text{p-up}}{\longrightarrow} (c,d)$, if $a = b \wedge c$ and $c \le d \le b \vee c$. Similarly, (a,b) is *prime-perspective down* to (c,d), in notation if $(a,b) \overset{\text{p-dn}}{\longrightarrow} (c,d)$, if $b = a \vee d$ and $a \wedge d \le c \le d$. If $(a,b),(c,d) \in L^2$, $a \le b$, $c \le d$, $b \wedge c = a$, and $b \vee c = d$, then[2] (a,b) is *up-perspective* to (c,d) and (c,d) is *down-perspective* to (a,b); the respective notations are $(a,b) \overset{\text{up}}{\sim} (c,d)$ and $(c,d) \overset{\text{dn}}{\sim} (a,b)$. If $(a,b) \overset{\text{up}}{\sim} (c,d)$, then $(a,b) \overset{\text{p-up}}{\longrightarrow} (c,d)$, and similarly for the "downward variant". Let N_5 denote the 5-element nonmodular lattice. For distinct covering pairs $(a,b),(c,d) \in \mathrm{Cp}(L)$, (the dual of) Note 1.2 of Grätzer [11] asserts that

$$\text{if } (a,b) \overset{\text{p-up}}{\longrightarrow} (c,d) \text{ but } (a,b) \overset{\text{up}}{\not\sim} (c,d), \text{ then } \{a,b,c,d,b \vee c\} \cong \mathsf{N}_5; \qquad (3.1)$$

in particular, $\{a,b,c,d,b \vee c\}$ is a sublattice of L. We recall the Prime Projectivity Lemma from Grätzer [11] in the following form.

Lemma 3.1 (Prime Projectivity Lemma, Grätzer [11])**.** *Let $(a,b),(c,d) \in \mathrm{Cp}(L)$ be distinct covering pairs of a finite lattice L. Then $\mathrm{con}(a,b) \ge \mathrm{con}(c,d)$ if and only if there is a finite sequence $(a,b) = (x_0,y_0)$, (x_1,y_1), $\dots$, $(x_n,y_n) = (c,d)$ of covering pairs of L such that for each $i \in \{1,\dots,n\}$, either $(x_{i-1},y_{i-1}) \overset{\text{p-up}}{\longrightarrow} (x_i,y_i)$ or $(x_{i-1},y_{i-1}) \overset{\text{p-dn}}{\longrightarrow} (x_i,y_i)$.*

Based on Grätzer [11] or the folklore, or trivially, note the following.

Lemma 3.2 (Grätzer [11])**.** *If $(a,b) = (x_0,y_0)$, (x_1,y_1), $\dots$, $(x_n,y_n) = (c,d)$ are pairs of elements of L such that, for each $i \in \{1,\dots,n\}$, either $(x_{i-1},y_{i-1}) \overset{\text{p-up}}{\longrightarrow} (x_i,y_i)$ or $(x_{i-1},y_{i-1}) \overset{\text{p-dn}}{\longrightarrow} (x_i,y_i)$, then $\mathrm{con}(c,d) \le \mathrm{con}(a,b)$.*

This paragraph, in the same way as Czédli [4], strengthens Theorem 3.10 from Grätzer [10]. A subset X of a poset P is a *down-set* if for every

[2]This definition of up-perspectivity is redundant; e.g., $b \wedge c = a$ implies that $a \le b$.

$u \in X$, the *order ideal* $\mathrm{idl}(u) := \{y \in P : y \le u\}$ is a subset of X. The collection $\mathrm{Dn}(P) = (\mathrm{Dn}(P); \subseteq)$ of all down-sets of P is a distributive lattice. Let L be a finite lattice. Since $\mathrm{Con}(L)$ is distributive, the structure theorem of finite distributive lattices gives that $\mathrm{Con}(L) \cong \mathrm{Dn}(\mathrm{Ji}(\mathrm{Con}(L)))$; see Theorems 107 and 149 of Grätzer [9]. Letting $\chi_L = \{\big((a,b),(c,d)\big) \in \mathrm{Cp}(L)^2 : \mathrm{con}(a,b) \le \mathrm{con}(c,d)\}$, $\big(\mathrm{Cp}(L); \chi_L\big)$ is a quasiordered set. We will write $(a,b) \le_{\chi_L} (c,d)$ instead of $\big((a,b),(c,d)\big) \in \chi_L$. It is well known, see Grätzer [11, 1st sentence], that $\{\mathrm{con}(a,b) : (a,b) \in \mathrm{Cp}(L)\}$ coincides with $\mathrm{Ji}(\mathrm{Con}(L))$. A subset W of $\mathrm{Cp}(L)$ is a *congruence-determining* subset of $\mathrm{Cp}(L)$ if $\{\mathrm{con}(a,b) : (a,b) \in W\}$ coincides with $\mathrm{Ji}(\mathrm{Con}(L))$. In this case, for brevity, we write $(W; \chi_L)$ instead of the more precise but lengthy $(W; \chi_L\rceil_W)$; it is a quasiordered set. A subset X of W is a χ_L*-down-set* of $(W; \chi_L)$ if for every $(a,b) \in X$ and $(c,d) \in W$, $(c,d) \le_{\chi_L} (a,b)$ implies that $(c,d) \in W$. The *collection of* χ_L*-down-sets* of $(W; \chi_L)$ will be denoted by $\mathrm{Dn}(W; \chi_L)$. Since $\mathrm{Cp}(L)$ is a congruence-determining subset of itself by Grätzer [11, 1st sentence], the $W = \mathrm{Cp}(L)$ particular case of the following lemma is the same as Grätzer [10, Theorem 3.10].

Lemma 3.3 (Czédli [3])**.** *If L is a finite lattice, then for every congruence-determining subset W of* $\mathrm{Cp}(L)$*, we have that* $\mathrm{Con}(L) \cong \mathrm{Dn}(W; \chi_L)$.

Since [3] provides only an outline rather than a proof and has not been published at the time of writing, we present an easy proof here.

Proof of Lemma 3.3. Let $J := \mathrm{Ji}(\mathrm{Con}(L))$. With reference to Theorems 107 and 149 of Grätzer [9], we have already mentioned that $\mathrm{Con}(L) \cong \mathrm{Dn}(J)$. Thus, it suffices to prove that $\mathrm{Dn}(W; \chi_L)$ and $\mathrm{Dn}(J)$ are isomorphic.

For $X \in \mathrm{Dn}(W; \chi_L)$, we define $f(X) := \{\mathrm{con}(a,b) : (a,b) \in X\}$. Since $J = \{\mathrm{con}(a,b) : (a,b) \in \mathrm{Cp}(L)\}$ (or since W is a congruence-determining subset of $\mathrm{Cp}(L)$), we have that $f(X) \subseteq J$. To show that $f(X) \in \mathrm{Dn}(J)$, assume that $\alpha \in f(X)$, $\beta \in J$, and $\beta \le \alpha$. As $\alpha \in f(X)$, $\alpha = f(a,b)$ for some $(a,b) \in X$. As W is a congruence-determining subset, $\beta = (c,d)$ for some $(c,d) \in W$. Since $X \in \mathrm{Dn}(W; \chi_L)$ and since $\mathrm{con}(c,d) = \beta \le \alpha = \mathrm{con}(a,b)$ means that $(c,d) \le_{\chi_L} (a,b)$, we obtain that $(c,d) \in X$. That is, $\beta = \mathrm{con}(c,d) \in f(X)$, showing that $f(X) \in \mathrm{Dn}(J)$. Consequently, $f \colon \mathrm{Dn}(W; \le_{\chi_L}) \to \mathrm{Dn}(J)$ is a map. Clearly, this map is order-preserving.

Conversely, we define a map $g \colon \mathrm{Dn}(J) \to \mathrm{Dn}(W; \le_{\chi_L})$ by letting, for $Y \in \mathrm{Dn}(J)$, $g(Y) := \{(a,b) \in W : \mathrm{con}(a,b) \in Y\}$. To show that $g(Y) \in \mathrm{Dn}(W; \le_{\chi_L})$, assume that $(a,b) \in g(Y)$, that is $\mathrm{con}(a,b) \in Y$, and $(c,d) \in W$ such that $(c,d) \le_{\chi_L} (a,b)$. Then $\mathrm{con}(c,d) \le \mathrm{con}(a,b)$ and $\mathrm{con}(a,b) \in Y \in \mathrm{Dn}(J)$ imply that $\mathrm{con}(c,d) \in Y$, whereby $(c,d) \in g(Y)$. Therefore,

$g(Y) \in \mathrm{Dn}(W; \leq_{\chi_L})$, g is indeed a map $g\colon \mathrm{Dn}(J) \to \mathrm{Dn}(W; \leq_{\chi_L})$, and this map is clearly order-preserving.

For $X \in \mathrm{Dn}(W; \chi_L)$, the inclusion $X \subseteq g(f(X))$ is obvious. Assume that $(a, b) \in g(f(X))$. Then $\mathrm{con}(a, b) \in f(X)$, that is, $\mathrm{con}(a, b) = \mathrm{con}(u, v)$ for some $(u, v) \in X$. Since $\mathrm{con}(a, b) = \mathrm{con}(u, v)$ yields that $(a, b) \leq_{\chi_L} (u, v)$, and since $X \in \mathrm{Dn}(W; \chi_L)$, we obtain that $(a, b) \in X$. Hence, $g(f(X)) \subseteq X$, and we have shown that $g \circ f$ is the identity map of $\mathrm{Dn}(W; \chi_L)$. Finally, let $Y \in \mathrm{Dn}(J)$. Using at $\overset{*}{=}$ that W is a congruence-determining subset, we obtain that

$$\begin{aligned} f(g(Y)) &= \{\mathrm{con}(a, b) : (a, b) \in g(Y)\} \\ &= \{\mathrm{con}(a, b) : (a, b) \in \{(u, v) \in W : \mathrm{con}(u, v) \in Y\}\} \\ &= \{\mathrm{con}(a, b) : (a, b) \in W \text{ and } \mathrm{con}(a, b) \in Y\} \overset{*}{=} Y. \end{aligned}$$

Thus, $f \circ g$ is the identity map of $\mathrm{Dn}(J)$, whereby f and g are reciprocal order isomorphisms, completing the proof of Lemma 3.3. □

Since the proof of the following well-known lemma is short and (the dual of) its idea will emerge later (see Case 2 in the proof of Lemma 4.4), we present the proof after stating the lemma.

Lemma 3.4 (Day [7, Page 71])**.** *For a finite lattice L and a subset W of $\mathrm{Cp}(L)$, if $\{(a^-, a) : a \in \mathrm{Ji}(L)\} \subseteq W$, then W is a congruence-determining subset of $\mathrm{Cp}(L)$.*

Proof. It suffices to show that $\{(a^-, a) : a \in \mathrm{Ji}(L)\}$ is a congruence-determining subset of $\mathrm{Cp}(L)$. Take a member of $\mathrm{Ji}(\mathrm{Con}(L))$; it is of the form $\mathrm{con}(u, v)$ with $(u, v) \in \mathrm{Cp}(L)$ by Grätzer [11, 1st sentence]. We can assume that $v \notin \mathrm{Ji}(L)$, since otherwise $u = v^-$ and there is nothing to show. Pick a minimal element $b \in \mathrm{idl}(v) \setminus \mathrm{idl}(u)$. Clearly, $b \in \mathrm{Ji}(L)$. Using that $b \parallel u$, $u \prec v$, $b^- \prec b$, $b^- \leq u \wedge b < b$, and $u < u \vee b \leq v$, we obtain that $(u, v) \overset{\mathrm{dn}}{\sim} (b^-, b)$ and $(b^-, b) \overset{\mathrm{up}}{\sim} (u, v)$. Thus, Lemma 3.2 yields that $\mathrm{con}(u, v) = \mathrm{con}(b^-, b) \in \{(a^-, a) : a \in \mathrm{Ji}(L)\}$. □

4 Further lemmas and completing the proof of Theorem 2.1

To enhance the paper's readability, we will present the proofs of the three parts of Theorem 2.1 in separate lemmas, with some parts needing multiple lemmas.

Lemma 4.1. *Let S be a cover-preserving sublattice of a finite lattice L. If there are subsets H and R of* $\mathrm{Cp}(L)$ *such that* $\{(a^{S,-},a) : a\in \mathrm{Ji}(S)\}\subseteq H\subseteq \mathrm{Cp}(S)$, $|R|\leq |L|-|S|$, $\{(a^{L,-},a) : a\in \mathrm{Ji}(L)\}\subseteq H\cup R$, *and* $H\cap R=\emptyset$, *then* $\mathrm{cd}(L)\leq \mathrm{cd}(S)$.

Proof. Since S is a cover-preserving sublattice, $\mathrm{Cp}(S)\subseteq \mathrm{Cp}(L)$. When dealing with the quasiordered sets $(H;\chi_L)$ and $(H;\chi_S)$, χ_L and χ_S will stand for the restrictions $\chi_L\rceil_H$ and $\chi_S\rceil_H$, respectively. Analogous conventions apply consistently throughout the paper. Let $W := H\cup R$. For $X\in \mathrm{Dn}(W;\chi_L)$, define $f(X) := X\cap H$. It follows from Lemma 3.1 that $\chi_L\rceil_H\supseteq \chi_S\rceil_H$, that is, $\chi_L\rceil_H$ is a *coarser relation* than $\chi_S\rceil_H$. Hence, $f(X)\in \mathrm{Dn}(H;\chi_S)$. So, $f\colon \mathrm{Dn}(W;\chi_L)\to \mathrm{Dn}(H;\chi_S)$ is a function. Let $Y\in \mathrm{Dn}(H;\chi_S)$ be a down-set within the range of f. Since W is the disjoint union of H and R, every f-preimage of Y has the unique form $X=Y\cup Z$, where $Z\subseteq R$. Thus, Y has at most $2^{|R|}$ preimages, and we obtain that

$$|\mathrm{Dn}(W;\chi_L)|\leq |\mathrm{Dn}(H;\chi_S)|\cdot 2^{|R|}. \tag{4.1}$$

By Lemma 3.4, W and H are congruence-preserving subsets of $\mathrm{Cp}(L)$ and $\mathrm{Cp}(S)$, respectively. Combining this fact with (4.1) and Lemma 3.3,

$$\begin{aligned}\mathrm{cd}(L) &= |\mathrm{Dn}(W;\chi_L)|/2^{|L|-1}\leq |\mathrm{Dn}(H;\chi_S)|\cdot 2^{|R|}/2^{|L|-1}\\ &\leq |\mathrm{Dn}(H;\chi_S)|\cdot 2^{|L|-|S|}/2^{|L|-1} = |\mathrm{Dn}(H;\chi_S)|/2^{|S|-1}=\mathrm{cd}(S). \qquad\square\end{aligned}$$

Lemma 4.2. *If S is a cover-preserving sublattice of a finite lattice L, then the inequality* $\mathrm{cd}(L)\leq \mathrm{cd}(S)$ *holds.*

Proof. First, we show that

$$\text{if } 0_S\neq a\in \mathrm{Ji}(L)\cap S, \text{ then } a\in \mathrm{Ji}(S) \text{ and } a^{L,-}=a^{S,-}\in S. \tag{4.2}$$

The membership $a\in \mathrm{Ji}(S)$ is trivial. Since $a^{S,-}\prec_S a$ and S is a cover-preserving sublattice, $a^{S,-}\prec_L a$, whereby $a^{L,-}=a^{S,-}\in S$, showing (4.2). There are two cases to consider, and each of them will be handled by applying Lemma 4.1.

First, we assume that $0_S=0_L$. Let $R := \{(a^{L,-},a) : a\in \mathrm{Ji}(L)\setminus S\}$ and $H := \mathrm{Cp}(S)$. Observing that $0_S=0_L\notin \mathrm{Ji}(L)$, (4.2) implies that $\{(a^{L,-},a) : a\in \mathrm{Ji}(L)\}\subseteq H\cup R$. Since $H\cap R=\emptyset$ and $|R|=|\mathrm{Ji}(L)\setminus S|\leq |L\setminus S|=|L|-|S|$ are clear, Lemma 4.1 yields the required $\mathrm{cd}(L)\leq \mathrm{cd}(S)$.

Second, we assume that $0_L<0_S$. Pick a lover cover $z\in L$ of 0_S. Let $R := \{(a^{L,-},a) : a\in \mathrm{Ji}(L)\setminus S\}\cup\{(z,0_S)\}$ and $H := \mathrm{Cp}(S)$. Regardless of whether $0_S\in \mathrm{Ji}(L)$, (4.2) implies that $\{(a^{L,-},a) : a\in \mathrm{Ji}(L)\}\subseteq H\cup R$. Since 0_L is neither in S nor in $\mathrm{Ji}(L)$, we have that $\mathrm{Ji}(L)\setminus S\subseteq L\setminus(\{0_L\}\cup S)$. Thus, $|\mathrm{Ji}(L)\setminus S|\leq |L|-1-|S|$, whereby $|R|\leq |L|-|S|$. These facts, the obvious $H\cap R=\emptyset$, and Lemma 4.1 imply the required $\mathrm{cd}(L)\leq \mathrm{cd}(S)$. $\square$

Lemma 4.3. *In a finite lattice K, let $a \prec b$ such that $a \in \mathrm{Mi}(K)$ and $b \in \mathrm{Ji}(K)$. Then for every $(x, y) \in \mathrm{Cp}(K)$, $(x, y) \leq_{\chi_K} (a, b) \iff (x, y) = (a, b)$.*

Proof. For $(u, v) \in \mathrm{Cp}(K) \setminus \{(a, b)\}$, $a \in \mathrm{Mi}(L)$ excludes that $(a, b) \overset{\text{p-up}}{\longrightarrow} (u, v)$, while $(a, b) \overset{\text{p-dn}}{\longrightarrow} (u, v)$ would contradict that $b \in \mathrm{Ji}(L)$. Thus, Lemma 3.1 applies, completing the proof. □

Lemma 4.4 (Edge Division Lemma)**.** *Let $a \prec b$ in a finite lattice K, and add a new element c to K such that $a \prec c \prec b$ in the new lattice $M := K \cup \{c\}$, c is doubly irreducible in M, and K is a sublattice of M. Then $\mathrm{cd}(M) \leq \mathrm{cd}(K)$.*

Proof. With $n := |K|$, we have that $|M| = n + 1$. We can assume that $n \geq 2$. Even though $\mathrm{Cp}(K)$ is not a subset of $\mathrm{Cp}(M)$, Lemmas 3.1 and 3.2 imply that for any $(x_1, y_1), (x_2, y_2) \in \mathrm{Cp}(K) \cap \mathrm{Cp}(M)$,

$$\text{if } (x_1, y_1) \leq_{\chi_K} (x_2, y_2), \text{ then } (x_1, y_1) \leq_{\chi_M} (x_2, y_2). \tag{4.3}$$

Similarly, these two lemmas imply that for any $(x_1, y_1), (x_2, y_2) \in \mathrm{Cp}(K)$

$$\text{if } (x_1, y_1) \leq_{\chi_K} (x_2, y_2), \text{ then } \mathrm{con}_M(x_1, y_1) \leq \mathrm{con}_M(x_2, y_2). \tag{4.4}$$

Depending on a and b, we consider two cases.

Case 1. In this case, we assume that $a \in \mathrm{Mi}(K)$ and $b \in \mathrm{Ji}(K)$. For $Y \in \mathrm{Dn}(\mathrm{Cp}(M); \chi_M)$, we define

$$f_1(Y) := \begin{cases} Y \setminus \{(c, b)\}, & \text{if } (a, c) \notin Y, \\ \{a, b\} \cup (Y \setminus \{(a, c), (c, b)\}), & \text{if } (a, c) \in Y. \end{cases} \tag{4.5}$$

We claim that $f_1(Y) \in \mathrm{Dn}(\mathrm{Cp}(K); \chi_K)$.

First, assume that $Y \in \mathrm{Dn}(\mathrm{Cp}(M); \chi_M)$ such that $(a, c) \notin Y$; then $f_1(Y)$ is computed by the first line of (4.5). Assume also that $(u, v) \in f_1(Y)$, $(x, y) \in \mathrm{Cp}(K)$, and $(x, y) \leq_{\chi_K} (u, v)$, that is, $\mathrm{con}_K(x, y) \leq \mathrm{con}_K(u, v)$. If $(u, v) = (a, b)$, then $(x, y) = (u, v) \in f_1(Y)$ by Lemma 4.3. Hence, we can assume that $(u, v) \neq (a, b)$. Then $(u, v) \in \mathrm{Cp}(K) \cap \mathrm{Cp}(M)$, and $(u, v) \in f_1(Y)$ together with (4.5) implies that $(u, v) \in Y$. If (x, y) is also in $\mathrm{Cp}(M)$, then $(x, y) \leq_{\chi_M} (u, v)$ by (4.3), whereby (x, y) is in the χ_M-down-set Y. Thus (x, y), which is distinct from $(c, b) \notin \mathrm{Cp}(K)$, belongs to $f_1(Y)$, as required. So we can assume that $(x, y) \in \mathrm{Cp}(K) \setminus \mathrm{Cp}(M)$, that is, $(x, y) = (a, b)$. By (4.4), $\mathrm{con}_M(a, b) = \mathrm{con}_M(x, y) \leq \mathrm{con}_M(u, v)$. Since the blocks of every lattice congruence are convex sublattices, $(a, c) \in \mathrm{con}_M(a, b)$, and so $\mathrm{con}_M(a,c) \le \mathrm{con}_M(a,b)$. Thus, by transitivity, $\mathrm{con}_M(a,c) \le \mathrm{con}_M(u,v)$, whereby $(a,c) \le_{\chi_M} (u,v) \in Y$, although Y is a χ_M-down-set and $(a,c) \notin Y$ has been assumed. This is a contradiction, which rules out the possibility that $(x,y) = (a,b)$ and implies that $f_1(Y) \in \mathrm{Dn}(\mathrm{Cp}(K); \chi_K)$.

Second, assume that $(a,c) \in Y \in \mathrm{Dn}(\mathrm{Cp}(M); \chi_M)$. Assume also that $(u,v) \in f_1(Y)$, $(x,y) \in \mathrm{Cp}(K) \setminus \{(u,v)\}$, and $(x,y) \le_{\chi_K} (u,v)$, that is, $\mathrm{con}_K(x,y) \le \mathrm{con}_K(u,v)$. Since $(a,b) \in f_1(Y)$ by (4.5), we can assume that $(x,y) \ne (a,b)$. By Lemma 4.3, $(u,v) \ne (a,b)$. Hence, both (x,y) and (u,v) are in $\mathrm{Cp}(M)$. Thus we obtain from (4.3) that $(x,y) \le_{\chi_M} (u,v)$. Since $(a,b) \ne (u,v) \in \mathrm{Cp}(K) \cap \mathrm{Cp}(M)$, (4.5) shows that $(u,v) \in Y$. From $(x,y) \le_{\chi_M} (u,v) \in Y$ and $Y \in \mathrm{Dn}(\mathrm{Cp}(M); \chi_M)$, we obtain that $(x,y) \in Y$. Combining $(x,y) \in Y$ with $(x,y) \in \mathrm{Cp}(K)$, $(a,c) \notin \mathrm{Cp}(K)$, and $(c,b) \notin \mathrm{Cp}(K)$, it follows that $(x,y) \in f_1(Y)$, as required.

We have shown that, regardless of whether (a,c) is in Y or not, $f_1(Y)$ belongs to $\mathrm{Dn}(\mathrm{Cp}(K); \chi_K)$. Thus, $f_1\colon \mathrm{Dn}(\mathrm{Cp}(M); \chi_M) \to \mathrm{Dn}(\mathrm{Cp}(K); \chi_K)$ is a map. We claim that

$$\text{each } X \in \mathrm{Dn}(K; \mathrm{Cp}(K)) \text{ has at most two} f_1\text{-preimages.} \tag{4.6}$$

To show this, first we note that $\mathrm{Cp}(M) \setminus \mathrm{Cp}(K) = \{(a,c),(c,b)\}$. Assume that $Y \in \mathrm{Dn}(\mathrm{Cp}(M); \chi_M)$ such that $f_1(Y) = X$. If $(a,b) \notin X$, then $X = f_1(Y)$ is computed by the first line of (4.5), whereby $Y \in \{X, X \cup (c,b)\}$, and X has at most two f_1-preimages. Similarly, if $(a,b) \in X$, then $f_1(Y)$ is determined by the second line of (4.5), and $Y \in \{(X \setminus \{(a,b)\}) \cup \{(a,c)\}, (X \setminus \{(a,b)\}) \cup \{(a,c),(c,b)\}\}$. Thus, X has at most two preimages again, proving (4.6).

It follows from (4.6) that $|\mathrm{Dn}(\mathrm{Cp}(M); \chi_M)| \le 2 \cdot |\mathrm{Dn}(\mathrm{Cp}(K); \chi_K)|$. Combining this inequality with Lemmas 3.3 and 3.4, we complete Case 1 by

$$\mathrm{cd}(M) = \frac{|\mathrm{Con}(M)|}{2^{(n+1)-1}} \le \frac{2 \cdot |\mathrm{Con}(K)|}{2^{(n+1)-1}} = \frac{|\mathrm{Con}(K)|}{2^{n-1}} = \mathrm{cd}(K). \tag{4.7}$$

Case 2. This case is devoted to the situation where $a \notin \mathrm{Mi}(K)$ or $b \notin \mathrm{Ji}(K)$. By duality, we can assume that $a \notin \mathrm{Mi}(K)$; see Figure 1, where the bold lines indicate coverings in M, while the thin solid line and the dotted line stand for "$<$" and "$\le$", respectively. The sole element of $M \setminus K$, namely c, is grey-filled. Since $a \ne \mathrm{Mi}(K)$ and $a \ne 1_K$ (as $a \prec_K b$), there is an element $d \in K$ such that $d \ne b$ and $a \prec_K d$. Let $u \in K$ be maximal element of $\mathrm{fil}(d) \setminus \mathrm{fil}(b)$, and define $v := b \vee u$. By the maximality of u, it is straightforward to obtain that $u \prec v$. Since $b \le u$ would contradict the choice of u and $u \le b$ would give that $d \le b$, contradicting that b and d are distinct covers of a, it follows that b and u are incomparable. So, using that

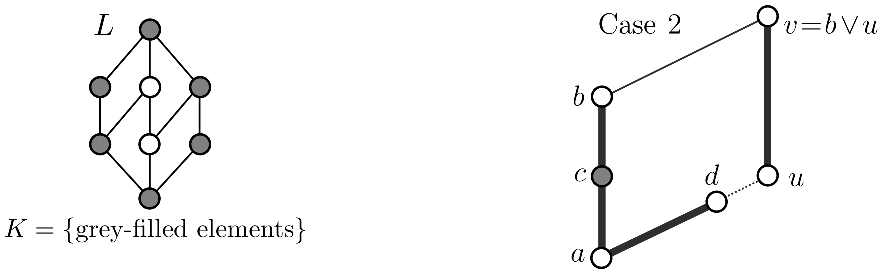

Figure 1: An example on the left and illustrating Case 2 in the proof of Lemma 4.4 on the right

$a \prec_K b$ and $a \le b \wedge u < b$, we have that $a = b \wedge u$. Hence, $(a,b) \overset{\text{up}}{\sim} (u,v)$ and $(u,v) \overset{\text{dn}}{\sim} (a,b)$. So, Lemma 3.1 yields that $\text{con}_K(a,b) = \text{con}_K(u,v)$. Combining this equality with Lemma 3.4, we obtain that

$$H := \text{Cp}(K) \setminus \{(a,b)\} \text{ is a congruence-determining subset of } \text{Cp}(K). \tag{4.8}$$

Turning our attention to M, observe that $c \not\le u$, since otherwise either $c = u \in K$ or $b = c^{M,+} \le u$ would be a contradiction. As $u \le c$ would also lead to a contradiction, namely $d \le u \le c \le b$, c and u are incomparable. Hence, $c < c \vee u$, and so $v = b \vee u = c^{M,+} \vee u \le (c \vee u) \vee u = c \vee u \le v$ gives that $c \vee u = v$, while $a \le c \wedge u \le b \wedge u = a$ implies that $c \wedge u = a$. Hence, $(a,c) \overset{\text{up}}{\sim} (u,v)$ and $(u,v) \overset{\text{dn}}{\sim} (a,c)$. Thus, $\text{con}_M(a,c) = \text{con}_M(u,v)$ by Lemma 3.1. This equality and Lemma 3.4 imply that

$$W := \text{Cp}(M) \setminus \{(a,c)\} \text{ is a congruence-determining subset of } \text{Cp}(M). \tag{4.9}$$

Note that $H = \text{Cp}(K) \cap \text{Cp}(M) \subseteq W$. For $Y \in \text{Dn}(W; \chi_M)$, let $f_2(Y) := Y \cap H$; we claim that $f_2(Y) \in \text{Dn}(H; \chi_K)$. Assume that $(p,q) \in f_2(Y)$, $(x,y) \in H$, and $(x,y) \le_{\chi_K} (p,q)$. As $H = \text{Cp}(K) \cap \text{Cp}(M)$, (4.3) yields that $(x,y) \le_{\chi_M} (p,q)$. Thus, using that $(p,q) \in f_2(Y) \subseteq Y \in \text{Dn}(W; \chi_M)$, we have that $(x,y) \in Y$, and so $(x,y) \in Y \cap H = f_2(Y)$. Therefore, the rule $f_2(Y) := Y \cap H$ defines a function $f_2 \colon \text{Dn}(W; \chi_M) \to \text{Dn}(H; \chi_K)$. Since $W \setminus H = \{(c,b)\}$ is a singleton, every $X \in \text{Dn}(H; \chi_K)$ has at most two f_2-preimages. Hence, $|\text{Dn}(W; \chi_M)| \le 2 \cdot |\text{Dn}(H; \chi_K)|$. Based on this inequality, (4.8), (4.9), and Lemma 3.3 imply the required $\text{cd}(M) \le \text{cd}(K)$ in the same way as in (4.7). This completes Case 2 and the proof of Lemma 4.4. □

Lemma 4.5. *If S is a sublattice of a finite modular lattice L, then* $\text{cd}(L) \le \text{cd}(S)$.

Proof. Throughout the proof, let S be a sublattice of a finite modular lattice L.

For $(a,b),(c,d) \in \mathrm{Cp}(L)$, we have that $(a,b) \overset{\mathrm{up}}{\sim} (c,d) \iff (c,d) \overset{\mathrm{dn}}{\sim} (a,b)$. We will refer to this fact as the *symmetry of perspectivity.* We know from Dedekind's criterion for modularity that a modular lattice cannot contain a sublattice isomorphic to N_5. Combining this fact with (3.1) and its dual, the symmetry of perspectivity, and Lemma 3.1, we obtain that for every $(x,y),(u,v) \in \mathrm{Cp}(L)$,

$$(x,y) \leq_{\chi_L} (u,v) \iff \mathrm{con}_L(x,y) = \mathrm{con}_L(u,v), \text{ so } \chi_L \text{ is an equivalence.} \tag{4.10}$$

Since S is also modular, χ_S is an equivalence relation on $\mathrm{Cp}(S)$, too.

We proceed by recalling a well-known isomorphism theorem for modular lattices (see, e.g., Grätzer [9, Theorem 348]), which asserts that perspective intervals in L are isomorphic (sub)lattices. That is, if $(a,b),(c,d) \in \mathrm{SO}(L)$ such that $(a,b) \overset{\mathrm{up}}{\sim} (c,d)$, then $g\colon [a,b] \to [c,d]$, defined by $x \mapsto c \vee x$, and $h\colon [c,d] \mapsto [a,b]$, defined by $y \mapsto b \wedge y$ are reciprocal lattice isomorphisms. Note that g and h are algebraic functions (univariate polynomials), whence they preserve congruence relations. Assume that $a \leq x \prec_L y \leq b$, and let $x' := g(x)$ and $y' := g(y)$. As g is a lattice isomorphisms, $c \leq x' \prec_L y' \leq d$. We claim that under the assumptions just established,

$$(x,y) \leq_{\chi_L} (x',y'),\ \ (x',y') \leq_{\chi_L} (x,y), \text{ and } \mathrm{con}_L(x,y) = \mathrm{con}_L(x',y'). \tag{4.11}$$

To see this, it suffices to deal with the congruences that the covering pairs in question generate. Since $(x,y) \in \mathrm{con}_L(x,y)$ and g preserves $\mathrm{con}_L(x,y)$, $(x',y') = (g(x),g(y)) \in \mathrm{con}_L(x,y)$, whereby $\mathrm{con}_L(x',y') \leq \mathrm{con}_L(x,y)$. Similarly, using that $h(x') = h(g(x)) = x$, $h(y') = y$, and h preserves $\mathrm{con}_L(x',y')$, we obtain that $\mathrm{con}_L(x,y) \leq \mathrm{con}_L(x',y')$, and we conclude (4.11). Note that (4.10) would allow a shorter formulation of (4.11).

Next, assume that $(a,b) = (x_0,y_0),(x_1,y_1),\dots,(x_t,y_t) = (c,d)$ is a sequence of distinct members of $\mathrm{SO}(L)$ such that for each $i \in \{1,\dots,t\}$, either $(x_{i-1},y_{i-1}) \overset{\mathrm{up}}{\sim} (x_i,y_i)$ or $(x_{i-1},y_{i-1}) \overset{\mathrm{dn}}{\sim} (x_i,y_i)$. (Since $(x_{i-1},y_{i-1}) \neq (x_i,y_i)$, $\overset{\mathrm{up}}{\sim}$ and $\overset{\mathrm{dn}}{\sim}$ cannot simultaneously hold.) For $i \in \{1,\dots,t\}$ such that $(x_{i-1},y_{i-1}) \overset{\mathrm{up}}{\sim} (x_i,y_i)$, let $g_i\colon [x_{i-1},y_{i-1}] \to [x_i,y_i]$ be the isomorphism defined by $g_i(\xi) := x_i \vee \xi$. Dually, for $i \in \{1,\dots,t\}$ such that $(x_{i-1},y_{i-1}) \overset{\mathrm{dn}}{\sim} (x_i,y_i)$, let $g_i\colon [x_{i-1},y_{i-1}] \to [x_i,y_i]$ be the isomorphism defined by $g_i(\xi) := y_i \wedge \xi$. Let $g\colon [a,b] \to [c,d]$ be the composite $g_t \dots g_2 g_1$ of the "stepwise" isomorphisms $g_t, \dots, g_1$. Applying (4.11) or its dual to each of these stepwise isomorphisms, we conclude that for all x and y satisfying $a \leq x \prec_L y \leq b$,

$$(x,y) \leq_{\chi_L} \bigl(g(x),g(y)\bigr) \text{ and } \mathrm{con}_L(x,y) = \mathrm{con}_L(g(x),g(y)). \tag{4.12}$$

We know from Lemma 4.2 that for the interval $[0_S,1_S]$ of L, we have that $\mathrm{cd}(L) \leq \mathrm{cd}([0_S,1_S])$. So, it suffices to prove that $\mathrm{cd}([0_S,1_S]) \leq \mathrm{cd}(S)$. In other words and to simplify the notation, we can assume that $L = [0_S, 1_S]$, that is, $0 := 0_S = 0_L$ and $1 := 1_L = 1_S$. Take a maximal chain $C = \{0 = c_0, c_1, \dots, c_{n-1}, c_n = 1\}$ in S such that $c_{i-1} \prec_S c_i$ for $i \in \{1, \dots, n\}$, and define $H_0 := \{(c_{i-1}, c_i) : i \in \{1, \dots, n\}\}$. Note that $H_0 = \mathrm{Cp}(C) \subseteq \mathrm{Cp}(S)$. We define

$$I := \{i : 1 \le i \le n \text{ and } (c_{k-1}, c_k) \le_{\chi_S} (c_{i-1}, c_i) \text{ holds for no } k < i\}, \tag{4.13}$$
$$H := \{(c_{i-1}, c_i) \in H_0 : i \in I\}. \tag{4.14}$$

In other words, H consists of the first members of the blocks of the equivalence $\chi_S\rceil_{H_0}$. We know from Grätzer and Nation [12] that H_0 is a congruence-determining subset of $\mathrm{Cp}(S)$. Hence, so is H by (4.10), (4.13), and (4.14). Furthermore, $\chi_S\rceil_H$ is the equality relation, whereby $\mathrm{Dn}(H; \chi_S)$ is the set of all subsets of H. Thus, denoting the powerset lattice $(\{X : X \subseteq H\}; \subseteq)$ of H by $\mathrm{Pow}(H)$, Lemma 3.3 yields that

$$\mathrm{Con}(S) \cong \mathrm{Dn}(H; \chi_S) = \mathrm{Pow}(H) \text{ and } |\mathrm{Con}(S)| = 2^{|H|} = 2^{|I|}. \tag{4.15}$$

Next, for $i \in I$, select $d_{i,0}, \dots, d_{i,m_i} \in L$ such that $d_{i,j-1} \prec_L d_{i,j}$ for $j \in \{1, \dots, m_i\}$, $d_{i,0} = c_{i-1}$, and $d_{i,m_i} = c_i$. For $k \in \{1, \dots, n\} \setminus I$, take the unique subscript $i(k)$ such that $k < i(k)$ and $(c_{k-1}, c_k) \le_{\chi_S} (c_{i(k)-1}, c_{i(k)})$. Combining Lemmas 3.1 and 3.2 with (4.12), we can fix an isomorphism $h_k \colon [c_{i(k)-1}, c_{i(k)}] \to [c_{k-1}, c_k]$ such that (4.12) remains valid when g is replaced by h_k. Let $m_k := m_{i(k)}$. We define $d_{k,0} := c_{k-1} = h_k(d_{i(k),0}) = h_k(c_{i(k)-1})$, $d_{k,1} := h_k(d_{i(k),1})$, $d_{k,2} := h_k(d_{i(k),2})$, $\dots$, $d_{k,m_k} =: h_k(d_{i(k),m_k}) = h_k(c_{i(k)})$. Since h_k is an isomorphism, $d_{k,j-1} \prec_L d_{k,j}$ for $j \in \{1, \dots, m_k\}$.

Let $W_0 := \{(d_{k,j-1}, d_{k,j}) : 1 \le k \le n \text{ and } 1 \le j \le m_k\}$. Since W_0 is the set of covering pairs of a maximal chain, W_0 is a congruence-determining subset of $\mathrm{Cp}(L)$ by Grätzer and Nation [12]. Let $W := \{(d_{i,j-1}, d_{i,j}) : i \in I \text{ and } 1 \le j \le m_i\}$. If $k \in \{1 \dots, n\} \setminus I$, then (4.12) with h_k replacing g yields that, for $j \in \{1, \dots, m_k\}$, $\mathrm{con}_L(d_{k,j-1}, d_{k,j}) = \mathrm{con}_L(d_{i(k),j-1}, d_{i(k),j})$. This fact, together with the fact that W_0 is a congruence-determining subset of $\mathrm{Cp}(L)$, implies that W is also a congruence-determining subset of $\mathrm{Cp}(L)$. Since $|\mathrm{Dn}(W; \chi_L)| \le 2^{|W|}$, Lemma 3.3 implies that $|\mathrm{Con}(L)| \le 2^{|W|}$.

Observe that $|W| = \sum_{i \in I} m_i$. For $i \in I$, the elements $d_{i,1}$, $d_{i,2}$, $\dots$, d_{i,m_i-1} are outside S. (If $m_i = 1$, then no such elements exist.) Furthermore, if $i, i' \in I$ such that $i < i'$, then $d_{i,1}$, $\dots$, d_{i,m_i-1} are smaller than—and therefore distinct from—each of $d_{i',1}$, $\dots$, $d_{i',m_{i'}-1}$. Hence, we obtain the second equality in the computation below from the fact that we take the union of pairwise disjoint subsets of $L \setminus S$:

$$|W| - |I| = \sum_{i \in I} (m_i - 1) = |\bigcup_{i \in I} \{d_{i,1}, \dots, d_{i,m_i-1}\}| \le |L| - |S|. \tag{4.16}$$

Finally, combining (4.15), (4.16), and the inequality $|\mathrm{Con}(L)| \le 2^{|W|}$ (proved above), we obtain that

$$\mathrm{cd}(L) = \frac{|\mathrm{Con}(L)|}{2^{|L|-1}} \le \frac{2^{|W|}}{2^{|L|-1}} = \frac{2^{|W|-|I|}}{2^{|L|-|S|}} \cdot \frac{2^{|I|}}{2^{|S|-1}} \le \frac{2^{|I|}}{2^{|S|-1}} = \frac{|\mathrm{Con}(S)|}{2^{|S|-1}} = \mathrm{cd}(S).$$

□

Proceeding to the final proof in the paper, we show that the lemmas established so far imply the theorem.

Proof of Theorem 2.1. Let S be a sublattice of a finite lattice L. If S is a cover-preserving sublattice of L or L is modular, then $\mathrm{cd}(L) \le \mathrm{cd}(S)$ by Lemma 4.2 or Lemma 4.5, respectively. Hence, we can assume that L is a dismantlable extension of S. Clearly, it suffices to deal with the particular case $|L| - |S| = 1$, which implies the general case by a trivial induction on $|L| - |S|$. So $S = L \setminus \{c\}$, where c is a doubly irreducible element of L. Let $a := \bigvee\{x \in S : x < c\}$ and $b := \bigwedge\{x \in S : x > c\}$. Since S is a sublattice, $\{a, b\} \subseteq S$ and $a <_L c <_L b$. If $a \prec_S b$, then, letting (S, c, L) play the role of (K, c, M), the (Edge Division) Lemma 4.4 yields the required inequality, $\mathrm{cd}(L) \le \mathrm{cd}(S)$.

Therefore, we can assume that b does not cover a in S. This assumption implies that $\mathrm{Cp}(L)$ is the disjoint union of $\mathrm{Cp}(S)$ and $\{(a, c), (c, b)\}$. Furthermore, $b \notin \mathrm{Ji}(L)$. Let $H := \mathrm{Cp}(S)$ and $R := \{(a, c)\} = \{(c^{L,-}, c)\}$. Observe that $|R| = 1 \le 1 = |L| - |S|$ and $H \cap R = \emptyset$. Since $b \notin \mathrm{Ji}(L)$, we have that $\{(x^{L,-}, x) : x \in \mathrm{Ji}(L))\} \subseteq H \cup R$. Therefore, the required $\mathrm{cd}(L) \le \mathrm{cd}(S)$ follows from Lemma 4.1, completing the proof of Theorem 2.1. □

References

[1] Kirby A. Baker, Peter C. Fishburn, and Fred S. Roberts: Partial Orders of Dimension 2. Networks 2 (1971), 11–28. `https://apps.dtic.mil/sti/pdfs/AD0707530.pdf`

[2] G. Czédli[3]: A note on finite lattices with many congruences. Acta Universitatis Matthiae Belii, Series Mathematics Online (2018), 22–28. `https://actamath.savbb.sk/pdf/aumb2602.pdf`

[3] G. Czédli: Lattices with many congruences are planar. Algebra Universalis (2019) 80:16. `https://doi.org/10.1007/s00012-019-0589-1`

[3]See https://www.math.u-szeged.hu/˜czedli/ for the author's preprints

[4] G. Czédli: The largest and all subsequent numbers of congruences of n-element lattices. https://tinyurl.com/czg-bsconl

[5] G. Czédli, L. Szabó: Quasiorders of lattices versus pairs of congruences. Acta Sci. Math. (Szeged) 60 (1995), 207–211. https://www.acta.hu/

[6] B. Davey: The product representation theorem for interlaced pre-bilattices: some historical remarks. Algebra Universalis 70 (2013) 403–409. https://doi.org/10.1007/s00012-013-0258-8

[7] A. Day: Characterizations of finite lattices that are bounded-homomorphic images or sublattices of free lattices. Canad. J. Math. 31, 69–78 (1979) https://doi.org/10.4153/CJM-1979-008-x

[8] R. Freese: Computing congruence lattices of finite lattices. Proc. Amer. Math. Soc. 125, 3457–3463 (1997) https://doi.org/10.1090/S0002-9939-97-04332-3

[9] G. Grätzer: Lattice Theory: Foundation. Birkhäuser Verlag, Basel (2011) https://doi.org/10.1007/978-3-0348-0018-1

[10] G. Grätzer: The Congruences of a Finite Lattice, A Proof-by-Picture Approach. Second edition. Basel: Birkhäuser, 2016. xxxii+347 p. https://doi.org/10.1007/978-3-031-29063-3.

[11] G. Grätzer: Congruences and prime-perspectivities in finite lattices. Algebra Universalis 74, 351–359 (2015) https://doi.org/10.1007/s00012-015-0355-y

[12] G. Grätzer, J. B. Nation: A new look at the Jordan-H́older theorem for semimodular lattices. Algebra Universalis 64, 309–311 (2010). https://doi.org/10.1007/s00012-011-0104-9

[13] C. Mureşan and J. Kulin: On the largest numbers of congruences of finite lattices. Order 37 (2020), 445–460 https://doi.org/10.1007/s11083-019-09514-2