ON CONJUGATELY SEPARABILITY OF NILPOTENT SUBGROUPS AND EQUATIONAL DOMAINS

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A group G is called CSA (conjugately separated abelian) if every maximal abelian subgroup of G is *malnormal*. This means that if H is a maximal abelian subgroup of G and $x \in G \setminus H$ then $H \cap H^x = 1$. The class of CSA groups is quite wide and has very serious roles in the study of residually free groups, universal theory of non-abelian free groups, limit groups, exponential groups and equational domains in algebraic geometry over groups (see [2], [3], [10] , and [11]). Another class of groups which has been studied extensively is the class of CT (commutative transitive) groups. A group is CT if commutativity is a transitive relation on the set of its non-identity elements. Despite this simple definition, the class of CT groups has also a crucial role in the study of residually free groups and so it has a close connection with CSA groups. Every CSA group is CT but the converse is not true. In the presence of residual freeness, both properties are equivalent, a theorem which is proved by B. Baumslag (see [1]).

During the past few decades, there have been many attempts to study these classes and their generalizations. A generalization of CT groups is introduced in [4] to extend the above mentioned theorem of B. Baumslag. Many interesting relations between CSA and CT groups are presented in [7] as well as an excellent account of the previous works.

It seems that the idea of CT and CSA groups is a small part of a very general concept. Suppose $\mathfrak{X}$ is a variety of groups (it can even be a universal class or even an inductive class of groups closed under subgroup). A group G can be called $\mathfrak{X}T$ then, if and only if for any two $\mathfrak{X}$-subgroups $K_1, K_2 \leq G$ the assumption $K_1 \cap K_2 \neq 1$ implies that $\langle K_1, K_2 \rangle$ is also an $\mathfrak{X}$-group. Similarly, we call a group G a $CS\mathfrak{X}$ group if all of its maximal $\mathfrak{X}$-subgroups are malnormal.

Although it seems that most parts of our work can be developed for many general classes $\mathfrak{X}$, we focus only on the variety of nilpotent groups of class at most k. Let's denote this variety by $\mathfrak{N}_k$. Hence, we call a group NT_k (nilpotency transitive of class k) if for any two $\mathfrak{N}_k$-subgroups K_1 and K_2, the assumption $K_1 \cap K_2 \neq 1$ implies that $\langle K_1, K_2\rangle$ is nilpotent of class at most k. Also a group G is CSN_k (conjugately separated nilpotent of class k) if and only if every maximal $\mathfrak{N}_k$-subgroup of G is malnormal. The case $k = 1$ obviously coincides with the ordinary CT and CSA groups. It is also easy to see that the property CSA implies CSN_k: this is true as in every non-abelian CSA group, solvable subgroups are abelian, so the maximal $\mathfrak{N}_k$-subgroups are automatically abelian. However, there is no implications of the form $\mathrm{NT}_k \to CT$ or $\mathrm{CSN}_k \to CSA$ (the second implication is not true as not every maximal abelian subgroup is necessarily a maximal $\mathfrak{N}_k$-subgroup). Despite this difference, we will show that these classes share many similar properties with the classical cases of CT and CSA groups. After a detailed study of basic properties of NT_k and CSN_k groups, as the main result, we will show that in presence of a special residuality condition, these two classes are the same. In order to do this, we will introduce a natural generalization of the concept of free group. Suppose A is a finitely generated free element of the variety $\mathfrak{N}_k$. Then every free product of a set of copies of A will be called an A-free group. Obviously, if $A = \mathbb{Z}$ then we get the ordinary free groups. We will show that in the class of residually A-free groups, the notions of NT_k and CSN_k groups are equivalent.

A group G is called an *equational domain* if and only if, for every positive integer n, the union of two algebraic sets in G^n is and algebraic set. In [2] it is shown that every non-abelian CSA group is an equational domain. We prove that this result is a special case of a general theorem: if a non-nilpotent group is CSN_k, then it is an equational domain. The same idea will be used to prove another generalization. We show that if a group G is not locally nilpotent but all of its maximal locally nilpotent subgroups are malnormal, then G is an equational domain.

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