Parameters in algebraically closed fields

Bruno Poizat[1]

Erlagol 2024

I begin with the description of a structure B that I call a *bifield,* which is definable in an algebraically closed field K ; I will explain later what rôle it plays in the Model Theory of Algebraically Closed Fields.

The underlying set of B is the union of two copies $L_1 \cup L_2$ of the base field K ; its language consists in the equivalence relation $x \in L_1 \leftrightarrow y \in L_2$, in the graphs of the two field operations (the ternary relations $x +_1 y = z \vee x +_2 y = z$, $x \times_1 y = z \vee x \times_2 y = z$) , and in an automorphism $\tau = (\tau_1,\tau_2)$ of B , where τ_1 is a field-isomorphism from L_1 to L_2 and τ_2 a field-isomorphism from L_2 to L_1 which are both definable in K .

In characteristic 0 , the identity is the only definable automorphism of the field K , so that τ_1 and τ_2 are inverses of each other : B is then nothing but a duplication of the base field K , and this case has no interest. Indeed, the quotient of B by the equivalence relation $x = \tau_2(y) \vee y = \tau_1(x)$ is a third copy L_3 of K which is definable without parameters in B .

In characteristic p , the definable automorphisms of K are the powers, positive or negative, of the Frobenius automorphism $x \rightarrow x^p$. Observe that $\tau_2(\tau_1(x)) = x^{p^\wedge n}$ implies that $\tau_1(\tau_2(y)) = y^{p^\wedge n}$, since every y has the form $\tau_1(x)$.

Lemma. (i) *If* n *is odd, the bifield* B *has no involutive automorphism.*
(ii) *If* n *is even,* B *has a unique involutive automorphism, which is a power of* τ *, and a third copy of the field* K *is definable without parameters in* B .

Proof. (i) I treat only the case n = 1 . Let σ be an automorphism of B whose square is the identity; if it preserves the two fields, by Artin's Theorem it acts on them as the identity. If it exchanges the two fields, it has the form $\sigma = (\sigma_1,\sigma_2)$, where σ_1 and σ_2 are inverses of each other; being an automorphism of B , it must commute with τ : $\tau_2\sigma_1 = \sigma_2\tau_1$, $\tau_1\sigma_2 = \sigma_1\tau_2$; hence $\tau_2\tau_1 = (\tau_2\sigma_1)^2$, which is impossible since the frobenius is not a square of a field automorphism.
(ii) Since the unique involutive automorphism of B is definable in it without parameters, we proceed as in characteristic 0 . **End**

Definition. An infinite structure S definable in an algebraically closed field K is said to be *autonomous* if any subset of a cartesian power of S , which is definable in the sense of K , is also definable by a formula (with parameters) of the language of S .

---

[1] Institut Camille Jordan, Université Claude Bernard (Lyon 1), 21 avenue du 8 mai 1945, 69622 Villeurbanne-cedex, France ; poizat@math.univ-lyon1.fr

Examples of autonomous structures
(i) a multifield (same as bifields, but with n fields instead of two);
(ii) a simple (infinite) algebraic group (model theoretic version of Borel-Tits Theorem);
(iii) a quasi-simple algebraic group.

Theorem. (i) *In characteristic* 0 *, an autonomous structure interprets without parameters a copy of the base field.*
(ii) *In characteristic* p *, an autonomous structure interprets without parameters a multifield of copies of the base field.*

Corollary. *Let* σ *be an automorphism of an autonomous structure.*
(i) *In characteristic* 0 *, if* σ *has a finite order,* σ^2 *is definable; if* (G,σ) *is superstable,* σ *is definable.*
(ii) *In characteristic* p *, if* σ *has a finite order, it is definable; if* σ *belongs to a superstable group of automorphisms of* G *, it is definable.*

The proofs rest on the basic model-theoretic properties of algebraically closed fields listed above; in particular, in showing the autonomy of simple algebraic groups (over an algebraically closed field), they use very few algebra:

- K eliminates the imaginary elements;

- any structure definable[2] in K is pseudo locally finite (based on compactness, elimination of imaginaries, and Galois theory of finite fields);

- K eliminates the quantifiers; the definable subsets of the cartesian powers of K are therefore the finite boolean combinations of Zariski-closed sets: they are called constructible sets in Geometry;

- any constructible group is constructibly isomorphic to an algebraic group (based on Weil's Theorem on group chunks);

- any infinite constructible field is constructibly isomorphic to the base field K (needs the preceding point);

- in a simple algebraic group G , a copy of the base field K is definable; one has to know that the Borel sugroups of G are not nilpotent, a very plain fact for a geometer; motivated model-theorists can obtain it by observing that a bad group cannot be pseudo locally finite;

- any simple infinite constructible group G is constructibly isomorphic to a Zariski-closed subgroup of some $Gl_n(K)$ defined by polynomial equations with integer coefficients; therefore, G has a constructible copy which is defined without parameters (helps to describe the theory of the group, which is ω_1-categorical by Zilber's Theorem).

Reference. B. Poizat, *Paramètres dans les corps algébriquement clos,* soumis.

[2] By "definable", we mean "definable with parameters", unless the converse is explicitely stated; we do not distinguish definability from interpretability.