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Positive Set Theory

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$\{ x \,/\, x \notin x \}$ does not exist, since the *collection axiom* $(\exists y)(\forall x)\ x \in y \Leftrightarrow x \notin x$ is inconsistent.

But what can be said of :

the empty set $a = \{ x \,/\, x \neq x \}$
the full set $c = \{ x \,/\, x = x \}$
the self-belongers' set $b = \{ x \,/\, x \in x \}$?

Without doubt: $a \notin a$, $b \notin a$, $c \notin a$, $a \in c$, $b \in c$, $c \in c$, $a \notin b$, $c \in b$ and a, b, c are distinct.

Questionable: $b \in b$ or $b \notin b$?

Initial model: $\{a, b, c\}^{i}$ where $b \notin b$.
Final model: $\{a, b, c\}^{f}$ where $b \in b$.

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Collectivising **formula: $(\exists\, y)(\forall\, x)\ \ x \in y \Leftrightarrow \varphi(x)$ is consistent ; set of formulae simultaneously collectivising.**

Positive **formula: $(\exists\, \overline{y})\ \varphi(x, \overline{y})$ where φ is boolean positive, possibly $\perp$, in the language of $=$ and $\in$.**

Thm 1. ***The positive formulae are simult. collectivising.***

Proof. $\{a, b, c\}^{f}$ satisfies $[(\exists\, y)\ y \in x] \Leftrightarrow x \in x$, so that every pos. formula is equivalent to $\perp$, or $x \in x$, or $x = x$.

Remark. In $\{a, b, c\}^{i}$ there is no $\{\, x \,/\, (\exists\, y)\ y \in x \,\}$.

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We note that $x \notin x$ is brutally negative, and we hope that Positive Logic will help us to select significant models.

h-inductive **axiom : $(\forall \bar{x})\ \varphi(\bar{x}) \Rightarrow \psi(\bar{x})$ where φ and ψ are positive (possibly $\perp$).**

Positively closed **model of an h-inductive theory T : for every homomorphism σ from M into another model N of T , $\bar{a}$ and $\sigma(\bar{a})$ satisfy the same positive formulae.**

The collection axioms are *not* h-inductive, by contrast to the witnessed c.a. $(\forall\ x)\ x \in a_\varphi \Leftrightarrow \varphi(x)$ when φ is pos.

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Assignation of witnesses to formulae must be injective, but their interpretations are not necessarily distinct. There is only one way to interpret witnesses on $\{a, b, c\}^f$.

Thm 2. $\{a, b, c\}^f$ ***is the unique positively closed model of the theory of witnessed collection for positive formulae.***

Proof. Let M be a model of the theory, in the language £ = { = , ∈ , ... a_φ , ... } ; we divide it into A = the empty sets, C = the points containing an empty set, B = the other points. When we send A to a , B to b , C to c , we obtain an £-homomorphism from M to $\{a, b, c\}^f$.

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The model $\{a, b, c\}^f$ has interesting properties, but also some defects: it contains no singleton, and we can doubt that in the real world the full set contains only three points; the remedy is to introduce witnesses for positive formulae with parameters in $\{a, b, c\}$, obtain a canonical model, and iterate the construction.

But this basic model gives the same witness to the formula $(\exists\ y)\ y \in x$ and to the formula $x \in x$, a condition which is untenable when we want to extend it.

Therefore we limit our ambitions and consider the theory of extensions of $\{a, b, c\}^f$ satisfying the collection axioms with witnesses for each *quantifier free* positive formula with parameters in $\{a, b, c\}$.

⑦

We can prove that this theory has a unique positively closed model, which is finite, and verifies the extensionality axiom.

In fact it satisfies $(\exists!\, y)(\forall x)\ \varphi(a, b, c, x) \Leftrightarrow x \in y$ for every boolean positive φ, and each of its points witnesses a boolean positive formula with parameters in $\{a, b, c\}$.

It satisfies also $x \in x \Rightarrow c \in x$, which is harmless, and even natural, but - alas - two conditions forbiding the iteration of the construction : $x \in x \Rightarrow a \in x \vee b \in x$ and $(\exists\, y)\ y \in x \Leftrightarrow a \in x \vee b \in x \vee c \in x$.

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We renounce to obtain our final model as a limit of a sequence of finite extensional structures, and we introduce witnesses in the following way: M_0 is the set of witnesses for the positive boolean formulae with parameters in $\{a, b, c\}^f$, ... M_{n+1} ... with parameters in M_n , ... Since there is no need to introduce different witnesses for obviously synonymous formulae, each M_n is finite.

The corresponding collection theory as a unique positively closed model M , which is extensional, and satisfies $(\forall x_1, \dots x_n)(\exists!\, y)(\forall x)\ \varphi(x_1, \dots x_n, x) \Leftrightarrow x \in y$ for every boolean positive φ ; each point of M is the witness of a positive boolean formula with parameters in M .

⑨

As far as combinatorics, or complexity, is concerned, M is equivalent to Arithmetic; with the help of formulae using negation, we can define in it the finite subsets of M with equicardinality, and also the hereditarily finite sets, and the finite ordinals (forming a definable part of M, not a set in the sense of M).

Conversely, the construction of M by induction can be represented in Arithmetic.

An open question is whether, following a similar line of arguments, more powerful models of positive set theory can be obtained: models containing a set for the natural integers, and a set for their subsets, etc.

⑩ More rudimentary models are obtained when we restrict the formulae. For instance, when we consider only non tautological bpf in the language of equality, the final model is made of the hereditarily finite sets.

In Positive Logic, there is nothing gratuitous in considering the contradiction $\bot$ as positive atomic; it is essential in Positive Model Theory, in Positive Sequent Calculus, and also in Positive Set Theory. Indeed, if we consider only pos. formulae free from $\bot$, the final model will have only one point c, satisfying $c \in c$; it is a model of the collection axiom for each positive non-$\bot$ formula, even with parameters; it satisfies, for every positive non-$\bot$ φ, $(\forall x_1, \ldots x_n)(\exists! y)(\forall x)\ \varphi(c, x_1, \ldots x_n, x) \Leftrightarrow x \in y$.