Groups of finite Morley rank with a generically multiply transitive action

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Introduction

- Joint work with A. Borovik.
- All structures have finite Morley rank, all subgroups and actions are definable, rk means Morley rank.
- Outline:

  1. Generically multiply transitive actions (2021?)
  2. Generically sharply multiply transitive actions (2018)
  3. Groups with a pseudoreflection subgroup (2012)
  4. Generic Identification Theorem for simple groups (2004, 2011)

Generic Transitivity

Assume that G is a group acting on a set X.

Definitions

If G is (sharply) transitive on a generic subset of X, then we say G acts generically (sharply) transitively on X. A definable subset $A \subseteq X$ is called generic in X if $\mathrm{rk}(A) = \mathrm{rk}(X)$.

Example

Let K be a field, then K^* acts generically sharply transitively on K^+, but not sharply transitively.

Assume that G is a group acting on a set X.

Definition

If the induced action of G on X^n is generically (sharply) transitive, then we say G acts generically (sharply) n-transitively on X.

Examples

For every $n \geqslant 1$, the natural action of:

- $\mathrm{GL}_n(K)$ on K^n is generically sharply n-transitive. (as a group!)
- $\mathrm{AGL}_n(K)$ on K^n is generically sharply $(n+1)$-transitive.
- $\mathrm{PGL}_{n+1}(K)$ on $\mathcal{P}_n(K)$ is generically sharply $(n+2)$-transitive.

Motivating Question

Problem (Borovik, Cherlin, 2008)

Let G be a connected group acting on a connected abelian group V definably, faithfully and generically n-transitively. If $n = \mathrm{rk}(V)$, then is it true that V has a vector space structure of dimension n over an algebraically closed field, and $G \cong \mathrm{GL}(V)$ acts on V naturally?

Remark

If true, then note that generically n-transitive and generically *sharply* n-transitive actions on abelian groups of rank n coincide.

Main Result

Let p be an odd prime.

Theorem (B., Borovik, 2021?)

Let G be a connected group acting on a connected elementary abelian p-group V definably, faithfully and generically n-transitively, where $n \geqslant \mathrm{rk}(V)$. Then $n = \mathrm{rk}(V)$ and $G \curvearrowright V$ is equivalent to $\mathrm{GL}_n(K) \curvearrowright K^n$ for some algebraically closed field K of characteristic p.

Small Cases

- $n = 1$ follows from Hrushovski and Zilber,
- $n = 2$ from Deloro (2009),
- $n = 3$ from Borovik, Deloro (2016), Frécon (2018).

Small Cases

Theorem (Deloro, 2009)

Let G be a connected non-solvable group acting faithfully on a connected abelian group V of rank 2. Then $G \curvearrowright V$ is equivalent to $\mathrm{GL}_2(K) \curvearrowright K^2$ or $\mathrm{SL}_2(K) \curvearrowright K^2$, for some algebraically closed field K.

Theorem (Borovik, Deloro, 2016) + (Frécon, 2018)

Let G be a connected non-solvable group acting faithfully and minimally on an abelian group V of rank 3. Then $G \curvearrowright V$ is equivalent to either the adjoint action $\mathrm{PSL}_2(K) \times Z(G) \curvearrowright K^3$, or the natural action $\mathrm{SL}_3(K) * Z(G) \curvearrowright K^3$ for some algebraically closed field K.

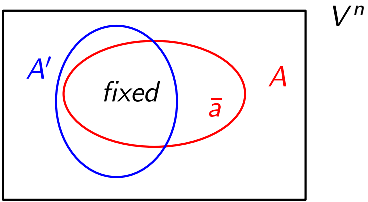
Case $n \geqslant 4$

Setting

G is a connected group, V is a connected elementary abelian p-group, where p is an odd prime, G acts on V definably, faithfully and generically n-transitively, and $n \geqslant \mathrm{rk}(V)$.

Observation

Let A be the generic subset of V^n on which G acts transitively. Then A is closed under taking inverses and permuting coordinates.



Hyperoctahedral Group

Observations

- Pick some $\bar{a} = (a_1, \ldots, a_n) \in A$. Then $(\pm a_1, \ldots, \pm a_n)$ and $\sigma(\bar{a}) = (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ lie in A.
- By transitivity, there exist an element $e_i \in G$ which maps $\bar{a}$ to $(a_1, \ldots, -a_i, \ldots, a_n)$ and $g_\sigma \in G$ which maps $\bar{a}$ to $\sigma(\bar{a})$ for $\sigma \in S_n$.
- Set H to be the setwise and N be the pointwise stabilizer in G of the set $\{\pm a_1, \ldots, \pm a_n\}$. Then $N \trianglelefteq H$ and the images of e_i's and g_σ's in H/N generate a subgroup isomorphic to $S_n \ltimes \mathbb{Z}_2^n$.
- A version of Maschke's Theorem and structural analysis of N shows that it is trivial.

Generically sharply n-transitive actions

Theorem (B., Borovik, 2018)

Let G be a connected group acting on a connected abelian group V definably, faithfully and generically sharply n-transitively, where $n \geqslant \mathrm{rk}(V)$. If V has no involutions, then $n = \mathrm{rk}(V)$ and $G \curvearrowright V$ is equivalent to $\mathrm{GL}_n(K) \curvearrowright K^n$ for some algebraically closed field K.

Sketch of Proof

- In this case, G contains copies of the hyperoctahedral group $S_n \ltimes \mathbb{Z}_2^n$.
- Set $U_i = [V, e_i]$ and assume without loss of generality that $\mathrm{rk}(U_i) = 1$, then $V = \oplus_{i=1}^n U_i$ and hence $\mathrm{rk}(V) = n$.
- Do induction on n; a subgroup of $C_G(e_1)$ acts generically sharply $(n-1)$-transitively on $\oplus_{i=2}^n U_i$ hence we have $\mathrm{GL}_{n-1}(K)$ in G.
- Obtain a torus $(K^*)^n$ of full rank in G where K is an algebraically closed field of odd or zero characteristic.

Pseudoreflection Subgroups

Theorem (B., Borovik, 2012)

Let G be a connected group acting on a connected abelian group V faithfully and irreducibly. If G contains a pseudoreflection subgroup R such that $\mathrm{rk}[V,R] = 1$, and $\mathrm{psrk}(G) = \mathrm{rk}(V)$, then $G \curvearrowright V$ is equivalent to $\mathrm{GL}_n(K) \curvearrowright K^n$ for some algebraically closed field K, where $n = \mathrm{rk}(V)$.

Definitions

A connected definable abelian subgroup R is called a pseudoreflection subgroup if $V = [V,R] \oplus C_V(R)$, and R acts transitively on the nonzero elements of $[V,R]$. Moreover, $\mathrm{psrk}(G)$ is the maximal number of pairwise commuting pseudoreflection subgroups in G.

Pseudoreflection Subgroups (Cont'd)

Theorem (B., Borovik, 2012)

Let G be a connected group acting on a connected abelian group V faithfully and irreducibly. If G contains a pseudoreflection subgroup R such that $\mathrm{rk}[V, R] = 1$, and $\mathrm{psrk}(G) = \mathrm{rk}(V)$, then $G \curvearrowright V$ is equivalent to $\mathrm{GL}_n(K) \curvearrowright K^n$ for some algebraically closed field K, where $n = \mathrm{rk}(V)$.

Sketch of Proof

Let G be a counter example of minimal Morley rank.

- Centralizers of non-central involutions in G are direct sums of general linear groups.
- G/Z is simple and the Generic Identification Theorem applies, hence G is a Chevalley group of Lie rank at least 3.
- We get a contradiction.

This finishes the proofs of both theorems.

Main Theorem: Characteristic 2

A Possible Theorem

Let G be a connected group acting on a connected elementary abelian 2-group V definably, faithfully and generically n-transitively, where $n \geqslant \mathrm{rk}(V)$. Then $n = \mathrm{rk}(V)$ and $(G, V) \cong (\mathrm{GL}_n(K), K^n)$ for some algebraically closed field K of characteristic 2.

Ingredients

- (Even Type Theorem) Infinite simple groups of finite Morley rank and even type are algebraic groups over algebraically closed fields. [Reference: Altınel, Borovik, Cherlin, Simple Groups of Finite Morley Rank, AMS, 2008. (xx+556 pp.)]
- Modules are rational. (More in Borovik's talk)

Main Theorem: Characteristic 0

Theorem (B., Borovik, 2021?)

Let G be a connected group acting on a connected torsion-free abelian group V definably, faithfully and generically n-transitively, where $n \geqslant \mathrm{rk}(V)$. Then $n = \mathrm{rk}(V)$ and $(G, V) \cong (\mathrm{GL}_n(K), K^n)$ for some algebraically closed field K of characteristic 0.

Reduction to Algebraic Groups

- (Loveys and Wagner, 1993) Let G be a connected group acting on a connected divisible abelian group V faithfully and minimally. Then $(G, V) \cong (H, K^n)$ for some algebraically closed field K of characteristic 0, and a subgroup $H \leqslant \mathrm{GL}(V)$.
- By generic n-transitivity, $\mathrm{rk}(G) \geqslant n^2$. Hence, $G \cong \mathrm{GL}(V)$.

Some Related Questions

Question 1 (B., Borovik, 2021)

Let G be a connected group acting on a (not necessarily abelian) connected group H definably, faithfully and generically 2-transitively. Is H abelian? What else can we say about H?

Question 2 (Borovik, Cherlin, 2008)

Let G be a connected group acting on a set X definably, faithfully and generically sharply $(n+2)$-transitively, where $n = \mathrm{rk}(X)$. Is it true that $(G, X) \cong (\mathrm{PGL}_{n+1}(K), \mathcal{P}_n(K))$, for some algebraically closed field K?

Tuna Altınel and Joshua Wiscons solved this problem for $n = 2$ and gave a partial result for $n \geqslant 3$.

Thanks

Spasibo

Rakhmet

Teșekkürler