# On definable closure in Hrushovski's strongly minimal generic structures

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# Introduction to strong minimality

A definable subset $D$ of a model of a first order theory $T$ is strongly minimal if in each $M \models T$ every definable subset of $D(M)$ is finite or cofinite. When $D$ is $x = x$, we say the model (theory) is strongly minimal.

In any such set algebraic closure gives a nice dependence relation, which implies the existence of a basis and an isomorphism between two models of a strongly minimal $T$ whose bases have the same cardinality.

Model theorists generalize the usual field theoretic notion by saying $a \in \mathrm{acl}(B)$ if for some formula $\phi(x, \bar{b})$ with $\bar{b} \in B$, $\phi(a, \bar{b})$ is true and $\phi(x, \bar{b})$ has only $n$ solutions for some $n < \omega$. If $n = 1$, $a \in \mathrm{dcl}(B)$.

# Zilber trichotomy

Zilber conjectured that all geometries of strongly minimal sets were

1. disintegrated (discrete/trivial) $\mathrm{acl}(A) = \bigcup_{a \in A} \mathrm{acl}(a)$ (e.g., $(\mathbb{Z}, S)$);
2. locally modular (vector space-like $e.g.$, $(\mathbb{Q}, +)$); or
3. field-like (e.g., $(\mathbb{C}, +, \times)$).

# The counterexample to Zilber's conjecture

- Hrushovski refuted this conjecture by a subtle extension of the Fraïssé construction. His *ab initio* (built from finite structures) examples, with *flat* geometries, have largely been treated as an inchoate collection of exotic structures because they admit no associative function (with infinite domain).

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- We amplify the divergence between algebraically closed fields and Hrushovski's original examples by showing that the latter admit no definable symmetric functions.

# Steiner system

- **Definition** A Steiner system with parameters $t, k, n$, written $S(t, k, n)$, is an $n$-element set $S$ together with a set of $k$-element subsets of $S$ (called blocks) with the property that each $t$-element subset of $S$ is contained in exactly one block.

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- Let $t = 2$. A block is called *a line*. So, any 2 points define a line. Each line consists of $k$ elements.
- For $t = 2$ and $n = \aleph_0$ there exist strongly minimal $k$-Steiner systems (John T. Baldwin and G. Paolini. Strongly Minimal Steiner Systems I. Journal of Symbolic Logic, 2020)

# Context

## Definition

1. The vocabulary $\tau$ contains a single ternary relation $R$. We require that $R$ is a predicate of 3-elements sets (distinct in any order).
2. Let $A$ be a finite structure in $\tau$. We write $R(A)$ for the collection of tuples $\bar{x}$ such that $A \models R(\bar{x})$.
3. We write $r(A)$ for the number of tuples (up to permutation) realizing $R$, i.e. $r(A) = |R(A)|/3!$.

# Predimension

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- (Steiner system) A linear space is a $\tau$-structure such that 2-points determine a unique line.
  We interpret $R$ as collinearity.
  For $B, \ell$ subsets of $A$, we say $\ell \in L(B)$ if $\ell$ is a maximal $R$-clique contained in $A$ and $|\ell \cap B| \geq 2$.
  Let

$$\delta(A) = |A| - \sum_{\ell \in L(A)} (|\ell| - 2).$$

# Definable closure

We introduce the (minimal) definable closure $\mathrm{dcl}^*$ of a set $X$ to distinguish points which depend on *all* elements of $X$.

Recall that for any first order theory $T$, if $X \subseteq M \models T$, then $c \in \mathrm{dcl}(X)$ implies the orbit of $c$ under $\mathrm{Aut}_X(M)$ ($X$ fixed pointwise) consists just of $c$.

The inverse holds in any $\omega$-homogenous model.
Also note that all models of a strongly minimal theory are $\omega$-homogeneous (Baldwin, Lachlan).

So, $\mathrm{dcl}(X)$ consists of those elements are fixed by $\mathrm{Aut}_X(M)$.

# Minimal definable closure

- By $b \in \mathrm{dcl}^*(X)$ we mean $b \in \mathrm{dcl}(X)$ but $b \notin \mathrm{dcl}(U)$ for any proper subset of $X$ (and analogously for $\mathrm{acl}^*$). Note that $\mathrm{dcl}^*(X)$ consists of elements of $\mathrm{dcl}(X)$ not fixed by $\mathrm{Aut}_T(M)$ for any $T \subsetneq X$.

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- The *symmetric definable closure* of $X$, $\mathrm{sdcl}(X)$, is those elements that are fixed by every $g \in \mathrm{Aut}_{\{X\}}(M)$ (the set of automorphisms of $M$ that fix $X$ setwise).

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- The *symmetric definable closure* of $X$, $\mathrm{sdcl}(X)$, is those elements that are fixed by every $g \in \mathrm{Aut}_{\{X\}}(M)$ (the set of automorphisms of $M$ that fix $X$ setwise).
- $b \in \mathrm{sdcl}^*(X)$ implies $b \in \mathrm{sdcl}(X)$ but $b \notin \mathrm{sdcl}(U)$ for any proper subset $U$ of $X$.

# Essentially unary functions

Let $T$ be a strongly minimal theory.

**Definition** An $\emptyset$-definable function $f(x_0 \ldots x_{n-1})$ is called *essentially unary* if there is an $\emptyset$-definable function $g(u)$ such that for some $i$, for all but a finite number of $c \in M$, and all but a set of Morley rank $< n$ of tuples $\bar{b} \in M^n$, $f(b_0 \ldots b_{i-1}, c, b_i \ldots b_{n-1}) = g(c)$.

# Truly $n$-ary functions

**Definition** Let $T$ be a strongly minimal theory.

- Let $\bar{x} = \langle x_0 \ldots x_{n-1} \rangle$: a function $f(\bar{x})$ *truly depends on* $x_i$ if $f(\bar{a}) \neq f(\bar{a}')$ for any *independent* sequence $\bar{a}$ and some (hence any) independent $\bar{a}'$ which disagrees with $\bar{a}'$ only in the $i$th place.

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- $f$ is truly $n$-ary if $f$ truly depends on all its arguments. and $f(\bar{a})$ is not a component of $\bar{a}$ for every $\bar{a}$ but a set of Morley rank less than $n$.

# Equivalent notions

## Lemma

For a strongly minimal $T$ the following conditions are equivalent:

1. $\mathrm{dcl}^*(I) = \emptyset$ for any independent set $I = \{a_1, a_2, \ldots a_n\}$ with $n > 1$;
2. every $\emptyset$-definable $n$-ary function ($n > 0$) is essentially unary;
3. for each $n > 1$ there is no $\emptyset$-definable truly $n$-ary function in $M$.

# The main result

## Theorem

Let $T_\mu$ be a Hrushovski construction or a strongly minimal Steiner system.

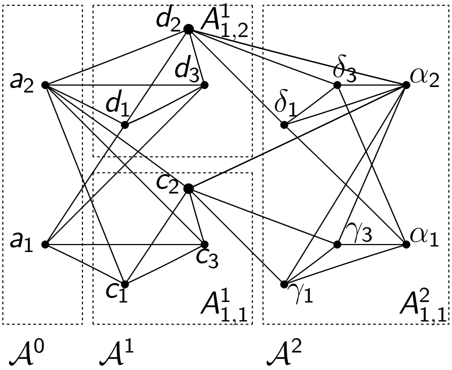
1. If $\delta(B) = 2$ implies $\mu(C/B) \geq 3$ for any good pair $C/B$, then $\mathrm{dcl}^*(I) = \emptyset$.
2. In any case $\mathrm{sdcl}^*(I) = \emptyset$.

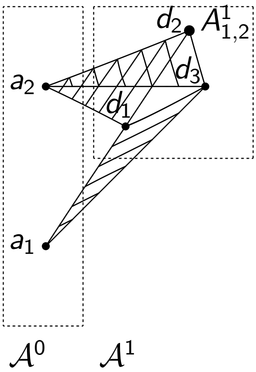
Consequently, $T_\mu$ does not admit elimination of imaginaries.

## Corollary

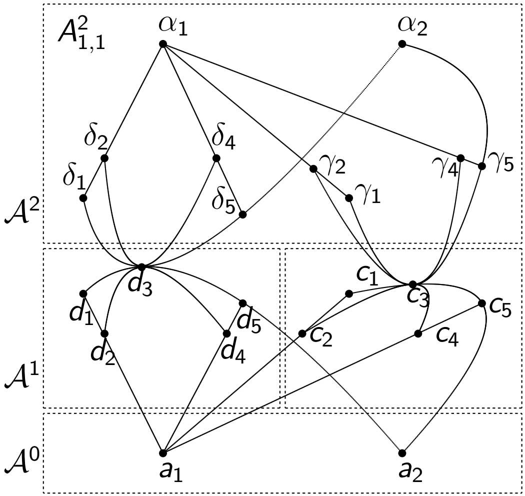
Assume $\delta(B) = 2$ implies $\mu(C/B) \geq 3$ for any good pair $C/B$, then $\mathrm{dcl}^*(I) = \emptyset$.
Then, for $n > 1$, no truly $n$-ary function is definable in $\hat{T}_\mu$ *even with parameters.*

# Counterexample for Hrushovski's example



# Counterexample for Steiner's system



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- In particular, B. Baizhanov asked whether any strongly minimal theory in a finite vocabulary, that admits elimination of imaginaries defines a field. We answer this question positively for the most evident counterexamples.
- But, the question of whether this can be extended when the class under consideration is expanded to arbitrary $\forall\exists$ classes of finite structures seems wide open.

# TOWARDS A FINER CLASSIFICATION OF STRONGLY MINIMAL SETS

1. **disintegrated geometry** For any $A$,

$$acl(A) = \bigcup_{a \in I} \mathrm{acl}(a);$$

# TOWARDS A FINER CLASSIFICATION OF STRONGLY MINIMAL SETS II

2. **strictly flat geometry** acl is not disintegrated but:
    1. $M$ is *dcl-disintegrated*: $\mathrm{dcl} I = \bigcup_{a \in I} \mathrm{dcl}(a)$ for independent $I$ (no $\emptyset$-definable truly $n$-ary functions);
    2. $M$ is not dcl-disintegrated: For some $n$ there are truly $n$-ary independent functions
        1. $M$ is *sdcl-disintegrated*: $\mathrm{sdcl} I = \bigcup_{a \in I} \mathrm{sdcl}(a)$ for independent $I$ (no commutative $\emptyset$-definable truly $n$-ary functions);
        2. $\emptyset$-definable binary functions with domain $M^2$ exist; e.g. quasigroups (J. Baldwin), and non-commutative counterexamples found here.
3. **Further examples**: ...

# Thank you

Thank you for attention!