Metrically Homogeneous Graphs and Distance Semigroups

Gregory Cherlin

THE STATE UNIVERSITY OF NEW JERSEY
RUTGERS

UNIVERSITÉ DE LYON
Université Claude Bernard Lyon 1

June 24, 2021
Universal Algebra and Model Theory

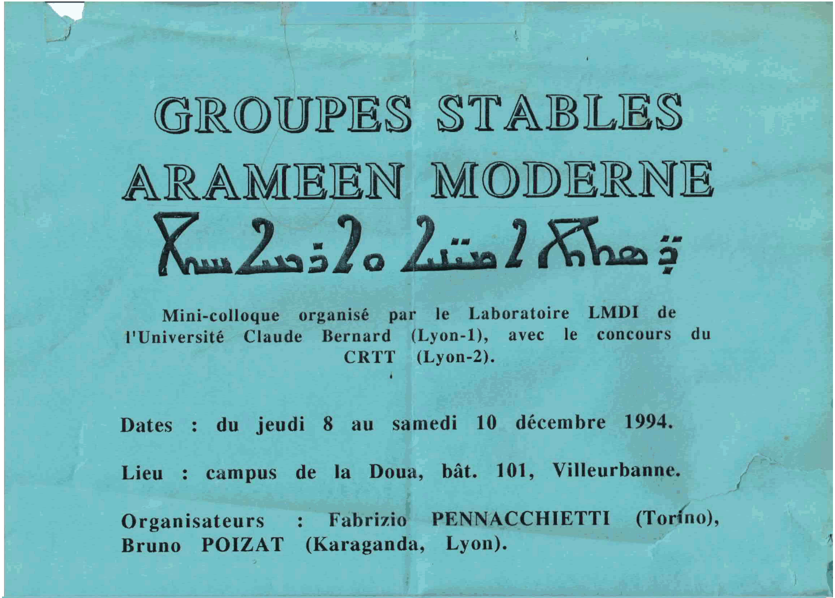
- Scattered introductory remarks ...
- Homogeneous Metric Spaces
- Metrically homogeneous graphs
- Distance Semigroups
- Toward a classification theorem

A Man of many Parts

Stable groups

Hiking and conversation

A Man of many Parts



GROUPES STABLES

ARAMEEN MODERNE

Mini-colloque organisé par le Laboratoire LMDI de l'Université Claude Bernard (Lyon-1), avec le concours du CRTT (Lyon-2).

Dates : du jeudi 8 au samedi 10 décembre 1994.

Lieu : campus de la Doua, bât. 101, Villeurbanne.

Organisateurs : Fabrizio PENNACCHIETTI (Torino), Bruno POIZAT (Karaganda, Lyon).

Stability and Algebra

William Ernest Marsh: On $\aleph_1$-categorical, not ω-categorical theories (1966).

Zilber ...

Groupe d'Étude de Théories Stables, 1977–1982—Bouscaren, Chatzidakis, Pillay, et al.

Lyon-1 ...

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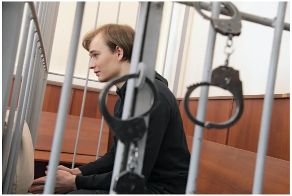
Lyon-1 . . .

Welcome home, Tuna—June 11, 2021

Azat Miftakhov



Azat Miftakhov Day Conference (June 16, 2021).
https://caseazatmiftakhov.org/2021/05/13/the-azat-miftakhov-day/

Introduction: Cédric Villani
Lectures by Maryna Viazovska, Alexander Bufetov, Peter Scholze, and words of support
Concluding remarks: Ilya Kapovich

n-point homogeneity

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Urysohn 1924, Birkhoff 1944, Fraïssé 1953, Freudenthal 1957

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Postulate III is not to be confused with the frequently stated weaker condition: "If M and N are isometric sets, then there exists a self-isometry of space which carries M into N." It should also be distinguished from the ***n-point homogeneity*** condition: "Any isometry between two sets of n or fewer points can be extended to a self-isometry of space." Thus Hilbert space has n-point homogeneity for every finite order n, yet does not satisfy the free mobility postulate; the same is true of Urysohn space (P. Urysohn, *Sur un espace métrique universel*, Bull. Sci. Math. vol. 51 (1927) pp. 43–64, 74–90).

Birkhoff 1944: Metric Foundations of Geometry

n-point homogeneity: displacement of rigid bodies
2-point homogeneity: displacement of rulers

2-Point homogeneity

Complete geodesic 2-point homogeneous spaces.
Compact: Wang 1951
Locally compact: Tits 1952 (Freudenthal 1957)

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Problem

Full classification for n-point homogeneous complete separable geodesic spaces?

A discrete analog . . .

The discrete setting

Integral metric spaces.

Geodesic paths.

2-point homogeneous: distance transitive graphs (with path metric)

n-point homogeneous: metrically homogeneous graphs

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Examples

n-gons

Icosahedron skeleton

Regular trees; the tree-like graphs $T_{m,n}$

(Dugald Macpherson)

The graphs $T_{m,n}$

Each point lies in a bouquet of m cliques of order n; no other cycles.

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Construction

Semi-regular tree with degrees m, n ($2 \leq m, n \leq \infty$)
Homogeneous as a bipartite graph with path metric.

Each part is a metrically homogeneous graph with rescaled metric (1/2): $T_{m,n}$ and $T_{n,m}$.

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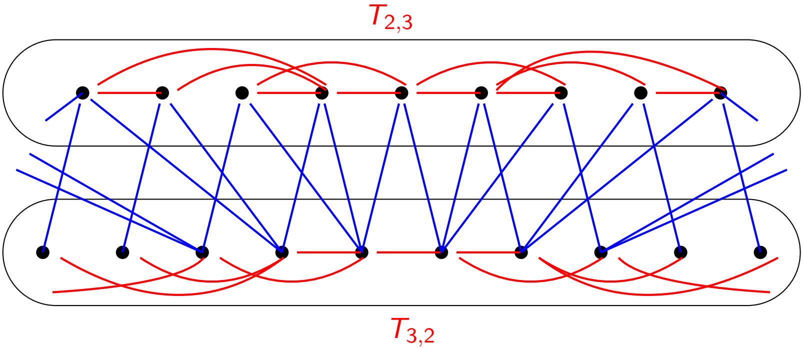

A catalog

The known metrically homogeneous graphs

- Finite (Cameron, 1980)
- Macpherson's graphs $T_{m,n}$
- Fraïssé limits $\lim_{\rightarrow} \mathcal{A}_3 \cap \mathcal{A}_H$

Fraïssé Constructions

Remark (Fraïssé)

Countable homogeneous metric spaces are determined up to isometry by their finite subspaces; equivalently, by their forbidden subspaces.

Example

The Fraïssé limit of the class of graphs is the random graph. The Fraïssé limit of the class of integral metric spaces is integral Urysohn space.

Notation (Finite combinatorics $\leftrightarrow$ Model theory)

$\mathcal{A} = \mathrm{Sub}(\Gamma)$; $\Gamma = \varinjlim \mathcal{A}$

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Definition

A metrically homogeneous graph is 3-constrained if its minimal forbidden metric subspaces have at most 3 points.

Fraïssé Constructions

Notation (Finite combinatorics $\leftrightarrow$ Model theory)

$\mathcal{A} = \mathrm{Sub}(\Gamma)$; $\Gamma = \lim_{\rightarrow} \mathcal{A}$

Theorem

The 3-constrained metrically homogeneous graphs are known. They form a 5-parameter family

$$\Gamma^{\delta}_{K_1,K_2,C_0,C_1}$$

where δ is the diameter, K_1, K_2 are the least and greatest k for which triangles of type $(k,k,1)$ are realized, and C_0, C_1 bound the perimeters of triangles of given parity.

Classification conjecture

Definition

A Henson constraint for a metrically homogeneous graph of diameter δ is a forbidden $(1, \delta)$-space.

Conjecture (Classification)

The infinite metrically homogeneous graphs of diameter δ are the Macpherson graphs $T_{m,n}$ and the graphs

$$\Gamma = \lim(\mathcal{A}_3 \cap \mathcal{A}_H)$$

where $\mathcal{A}_3$ is associated with a 3-constrained graph of diameter δ and $\mathcal{A}_H$ is a class determined by Henson constraints.

Topological dynamics

Theorem (Prague consortium)

Most of the known metrically homogeneous graphs of generic type are characterized by a finite set of homomorphically forbidden partial subspaces.
Hence the automorphism group has a metrizable universal minimal flow.

(Exceptions: imprimitive graphs, e.g. bipartite: there are infinitely many odd cycles to be forbidden.)

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(Exceptions: imprimitive graphs, e.g. bipartite: there are infinitely many odd cycles to be forbidden.)
The finiteness theorem involves a new theory of generalized metric spaces with values in an appropriate commutative semigroup.

Generalized metric spaces

Definition

Let $(S, +, \prec)$ be a semigroup equipped with a partial order. An S-valued metric space (Γ, d) consists of a set of points Γ and a symmetric S-valued function $d(x, y)$ defined for $x \neq y$ satisfying the triangle inequality in the form

$$d(x, y) \preceq d(x, z) + d(y, z)$$

A partial S-valued metric space is a graph with edge labels in S which has a completion to an S-valued metric space.

Path completion

Definition

Let $(S, +, \prec)$ be a commutative semigroup with a partial order, and A a finite graph with edge labels in S. The path metric on A is the partial function $d(x, y)$ defined for $x \neq y$ as

$$d(x, y) = \inf(||\gamma|| \mid \gamma \text{ a path from } x \text{ to } y)$$

if this infimum exists with respect to $\preceq$.

Distance semigroups

Definition (Tentative)

Let $(S, +, \prec)$ be a commutative semigroup with a partial order $\prec$. We say that the structure is a distance semigroup if for every partial S-valued metric space the corresponding path metric $d(x, y)$ is defined for all x, y distinct.
A distance semigroup is special if the Fraïssé limit of the S-valued metric spaces exists.

We can also define a similar notion without assuming associativity (path weights become multi-valued).

Example: Sauer

Example (Sauer)

Let $S \subseteq \mathbb{R}$ be finite and define

$$a +_S b = \max(s \in S \mid s \leq a + b)$$

with the usual order.

Then the S-valued metric spaces are the usual metric spaces with values in S.

Then the following are equivalent:

- The S-valued metric spaces have a Fraïssé limit.
- $(S, +_S)$ is a semigroup.
- $(S, +_S, <)$ is a special distance semigroup.

Example: Braunfeld

Example (Sam Braunfeld)

Let S be a finite lattice viewed as a semigroup with operation $\vee$ and the corresponding partial order.

Then S is certainly a distance semigroup and is special if and only if S is distributive.

Example: Hubička, Konečný, Nešetřil

Given δ, M, C with $C > 2\delta$ and $\delta/2 \leq M \leq (C - \delta - 1)/2$ define $D^{\delta}_{M,C}$ as follows. The underlying set is $[\delta] = (1, \ldots, \delta)$. The addition operation $i +_{M,C} j$ is defined as the point in the interval $[d^-, d^+]$ closest to M, where

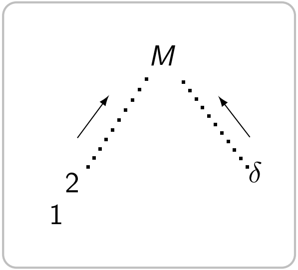
$$d^- = |i - j|$$
$$d^+ = \min(i + j, C - 1 - (i + j))$$

The partial order $i \prec j$ is either the natural order

$$j = i +_{M,C} x \text{ for some } x,$$

or its extension by the condition $i \prec M - 1$ ($i \neq M - 1, M$) when $C = 2\delta + M$.
Then $([\delta], +_{M,C}, \prec)$ is a distance semigroup with maximum element M.

$D^{\delta}_{M,C}$



$D_{M,C}$
$(M + x = M)$

"Cross-relations" not shown. E.g. $1 \leq_{M,C} \delta - 1$.

An application

Theorem (Prague)

If Γ is a primitive metrically homogeneous graph of known type with diameter δ and $C = \min(C_0, C_1)$, then for some M Γ satisfies the $D^{\delta}_{M,C}$-triangle inequality

$$d(x, y) \leq_{M,C} d(x, z) +_{M,C} d(z, y)$$

and shortest path completion is a completion procedure for finite partial subspaces of Γ.

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$$K_1 \leq M \leq K_2$$

We take $\prec$ to be the natural order unless $M = K_1$ and $C = 2\delta + K_1$.

Another application

Lemma

Let Γ be a primitive metrically homogeneous graph of known type, and $d \geq \max(K_1, 2)$ fixed. Then the class $\mathcal{A}^d$ of subspaces of Γ of diameter at most d has a Fraïssé limit.

For the proof, if $d \geq \delta/2$ then shortest path completion with respect to $M = \max(K_1, \lceil \delta/2 \rceil)$ gives the claim. If $d < \delta/2$ then also $K_1 < \delta/2$ so the first step allows us to reduce the diameter of Γ and repeat the argument.

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This leads to a projected strategy to prove the classification theorem by a series of inductive arguments.

Toward the classification

Work in progress with Amato.
Γ infinite, not of the form $T_{m,n}$.

- Define $\delta, K_1, K_2, C_0, C_1$ and H.
- Show these parameters control the triangles of Γ as expected.
- Show the Fraïssé limit Γ^* exists.
- It follows that any finite subspace of Γ embeds isometrically into Γ^*.
- Prove the converse: if A is a finite subspace of Γ^* then A embeds into Γ. Proceed by induction on the diameter d of A, for $d \geq \max(K_1, 2)$.

As Γ is determined by its finite subspaces, it follows that Γ is isometric to Γ^*.

Problems

- Understand distance semigroups and special distance semigroups. Is there a purely algebraic definition? To what extent can the partial order be varied?
- Some form of the following distributive law appears relevant (where defined).

$$x + \inf S = \inf(x + S)$$

- Is associativity in fact an assumption, or a conclusion?
- Proof of the existence of the relevant Fraïssé limits (amalgamation) is complicated. There should be a direct proof by shortest path completion.
- Relate the algebra of distance semigroups to the model theory of homogeneous binary structures in symmetric languages.

We have mainly examples at this point.