CHARACTERISTICS FOR FAMILIES OF THEORIES

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Erlagol-2021
June 27, 2021

Ranks for families of theories

Let Σ be a language. If Σ is relational we denote by $\mathcal{T}_\Sigma$ the family of all theories of the language Σ. If Σ contains functional symbols f then $\mathcal{T}_\Sigma$ is the family of all theories of the language Σ', which is obtained by replacements of all n-ary symbols f with $(n+1)$-ary predicate symbols R_f interpreted by $R_f = \{(\bar{a}, b) \mid f(\bar{a}) = b\}$.
By $F(\Sigma)$ we denote the set of all formulas in the language Σ and by $\mathrm{Sent}(\Sigma)$ the set of all sentences in $F(\Sigma)$.
For a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ we denote by $\mathcal{T}_\varphi$ the set of all theories $T \in \mathcal{T}$ with $\varphi \in T$.

Any set $\mathcal{T}_\varphi$ is called the *φ-neighbourhood*, or simply a *neighbourhood*, for $\mathcal{T}$, or the (φ-)*definable* subset of $\mathcal{T}$. The set $\mathcal{T}_\varphi$ is also called (*formula*- or *sentence*-)*definable* (by the sentence φ) with respect to $\mathcal{T}$, or (*sentence*-)*$\mathcal{T}$-definable*, or simply *s-definable*. We define the *rank* $\mathrm{RS}(\cdot)$ for families $\mathcal{T} \subseteq \mathcal{T}_\Sigma$, similar to Morley rank for a fixed theory, and a hierarchy with respect to these ranks in the following way.

Definition[1]. For the empty family $\mathcal{T}$ we put the rank $\mathrm{RS}(\mathcal{T}) = -1$, for finite nonempty families $\mathcal{T}$ we put $\mathrm{RS}(\mathcal{T}) = 0$, and for infinite families $\mathcal{T}$ — $\mathrm{RS}(\mathcal{T}) \geq 1$.
For a family $\mathcal{T}$ and an ordinal $\alpha = \beta + 1$ we put $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if there are pairwise inconsistent $\Sigma(\mathcal{T})$-sentences φ_n, $n \in \omega$, such that $\mathrm{RS}(\mathcal{T}_{\varphi_n}) \geq \beta$, $n \in \omega$.
If α is a limit ordinal then $\mathrm{RS}(\mathcal{T}) \geq \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \beta$ for any $\beta < \alpha$.
We set $\mathrm{RS}(\mathcal{T}) = \alpha$ if $\mathrm{RS}(\mathcal{T}) \geq \alpha$ and $\mathrm{RS}(\mathcal{T}) \not\geq \alpha + 1$.
If $\mathrm{RS}(\mathcal{T}) \geq \alpha$ for any α, we put $\mathrm{RS}(\mathcal{T}) = \infty$.

[1] Sudoplatov S. V. Ranks for families of theories and their spectra // Lobachevskii Journal of Mathematics (to appear). arXiv:1901.08464v1 [math.LO]. — 2019. — 17 p.

Totally transcendental families

A family $\mathcal{T}$ is called *e-totally transcendental*, or *totally transcendental*, if $\mathrm{RS}(\mathcal{T})$ is an ordinal.
If $\mathcal{T}$ is *e*-totally transcendental, with $\mathrm{RS}(\mathcal{T}) = \alpha \geq 0$, we define the *degree* $\mathrm{ds}(\mathcal{T})$ of $\mathcal{T}$ as the maximal number of pairwise inconsistent sentences φ_i such that $\mathrm{RS}(\mathcal{T}_{\varphi_i}) = \alpha$.

e-minimal families

Definition[2]. An infinite family $\mathcal{T}$ is called *e-minimal* if for any sentence $\varphi \in \Sigma(\mathcal{T})$, $\mathcal{T}_\varphi$ is finite or $\mathcal{T}_{\neg\varphi}$ is finite.

By the definition a family $\mathcal{T}$ is *e*-minimal iff $\mathrm{RS}(\mathcal{T}) = 1$ and $\mathrm{ds}(\mathcal{T}) = 1$, and iff $\mathcal{T}$ has unique accumulation point.

[2] Sudoplatov S. V. Approximations of theories // Siberian Electronic Mathematical Reports. — 2020. — Vol. 17. — P. 715–725.

E-closures

In the paper [Sudoplatov S. V. Closures and generating sets related to combinations of structures / S. V. Sudoplatov // Bulletin of Irkutsk State University. Series "Mathematics". — 2016. — Vol. 16. — P. 131–144.] the notion of E-closure was introduced and characterized as follows:

Proposition 1

If $\mathcal{T} \subseteq \mathcal{T}_\Sigma$ is an infinite set and $T \in \mathcal{T}_\Sigma \setminus \mathcal{T}$ then $T \in \mathrm{Cl}_E(\mathcal{T})$ (i.e., T is an *accumulation point* for $\mathcal{T}$ with respect to E-closure Cl_E) if and only if for any sentence $\varphi \in T$ the set $\mathcal{T}_\varphi$ is infinite.

The following theorem characterizes the property of *e*-total transcendency for countable languages.

Theorem 1

For any family $\mathcal{T}$ with $|\Sigma(\mathcal{T})| \leq \omega$ the following conditions are equivalent:
(1) $|\mathrm{Cl}_E(\mathcal{T})| = 2^\omega$;
(2) $e\text{-}\mathrm{Sp}(\mathcal{T}) = 2^\omega$;
(3) $\mathrm{RS}(\mathcal{T}) = \infty$;
(4) there exists a 2-tree of sentences φ for s-definable properties $\mathcal{T}_\varphi$.

[3] Sudoplatov S. V. Ranks for families of theories and their spectra // Lobachevskii Journal of Mathematics (to appear). arXiv:1901.08464v1 [math.LO]. — 2019. — 17 p.

[4] Pavlyuk In. I., Sudoplatov S. V. Formulas and properties for families of theories of Abelian groups // Bulletin of Irkutsk State University. Series Mathematics. — 2021. — Vol. 36.

Ranks for families of theories[5]

Theorem 2

For any language Σ either $\mathrm{RS}(\mathcal{T}_\Sigma)$ is finite, if Σ consists of finitely many 0-ary and unary predicates, and finitely many constant symbols, or $\mathrm{RS}(\mathcal{T}_\Sigma) = \infty$, otherwise.

[5] Markhabatov N. D., Sudoplatov S. V. Ranks for families of all theories of given languages // Eurasian Mathematical Journal. — 2021. — Vol. 12, No. 2.

Ranks for families of theories[6]

For a language Σ we denote by $\mathcal{T}_{\Sigma,n}$ the family of all theories in $\mathcal{T}_\Sigma$ having n-element models, $n \in \omega$, as well as by $\mathcal{T}_{\Sigma,\infty}$ the family of all theories in $\mathcal{T}_\Sigma$ having infinite models.

Theorem 3

For any language Σ either $\mathrm{RS}(\mathcal{T}_{\Sigma,n}) = 0$, if Σ is finite or $n = 1$ and Σ has finitely many predicate symbols, or $\mathrm{RS}(\mathcal{T}_{\Sigma,n}) = \infty$, otherwise.

Theorem 4

For any language Σ either $\mathrm{RS}(\mathcal{T}_{\Sigma,\infty})$ is finite, if Σ is finite and without predicate symbols of arities $m \geq 2$ as well as without functional symbols of arities $n \geq 1$, or $\mathrm{RS}(\mathcal{T}_{\Sigma,\infty}) = \infty$, otherwise.

[6] Markhabatov N. D., Sudoplatov S. V. Ranks for families of all theories of given languages // Eurasian Mathematical Journal. — 2021. — Vol. 12, No. 2.

Ranks for families of theories

Proposition 2

For any language Σ either either $\mathcal{T}_\Sigma$ is countable, if if Σ consists of finitely many 0-ary and unary predicates, and finitely many constant symbols, or $|\mathcal{T}_\Sigma| \geq 2^\omega$, otherwise.

Proposition 3

For any language Σ either $\mathcal{T}_{\Sigma,n}$ is finite, if Σ is finite or $n = 1$ and Σ has finitely many predicate symbols, or $|\mathcal{T}_{\Sigma,n}| \geq 2^\omega$, otherwise.

Proposition 4

For any language Σ either $\mathcal{T}_{\Sigma,\infty}$ is at most countable, if Σ is finite and without predicate symbols of arities $m \geq 2$ as well as without functional symbols of arities $n \geq 1$, or $|\mathcal{T}_{\Sigma,\infty}| \geq 2^\omega$, otherwise.

Definable subfamilies[7]

Definition. If $\mathcal{T}$ is a family of theories and Φ is a set of sentences, then we put $\mathcal{T}_\Phi = \bigcap\limits_{\varphi\in\Phi} \mathcal{T}_\varphi$ and the set $\mathcal{T}_\Phi$ is called (*type*-) or (*diagram-*)*definable* (by the set Φ) with respect to $\mathcal{T}$, or (*diagram-*)*$\mathcal{T}$-definable*, or simply *d-definable*.
Clearly, finite unions of d-definable sets are again d-definable. Considering infinite unions $\mathcal{T}'$ of d-definable sets $\mathcal{T}_{\Phi_i}$, $i \in I$, one can represent them by sets of sentences with infinite disjunctions $\bigvee\limits_{i\in I} \varphi_i$, $\varphi_i \in \Phi_i$. We call these unions $\mathcal{T}'$ are called *d_∞-definable* sets.

[7] Markhabatov N. D., Sudoplatov S. V. Definable subfamilies of theories, related calculi and ranks // Siberian Electronic Mathematical Reports. — 2020. — Vol. 17. — P. 700–714.

Definable subfamilies[8]

Definition. Let $\mathcal{T}$ be a family of theories, Φ be a set of sentences, α be an ordinal $\leq \mathrm{RS}(\mathcal{T})$ or -1. The set Φ is called *α-ranking* for $\mathcal{T}$ if $\mathrm{RS}(\mathcal{T}_\Phi) = \alpha$. A sentence φ is called *α-ranking* for $\mathcal{T}$ if $\{\varphi\}$ is *α-ranking* for $\mathcal{T}$.
The set Φ (the sentence φ) is called *ranking* for $\mathcal{T}$ if it is α-ranking for $\mathcal{T}$ with some α.

[8] Markhabatov N. D., Sudoplatov S. V. Definable subfamilies of theories, related calculi and ranks // Siberian Electronic Mathematical Reports. — 2020. — Vol. 17. — P. 700–714.

Definable subfamilies and ranks[9]

Proposition 5

For any ordinals $\alpha \leq \beta$, if $\mathrm{RS}(\mathcal{T}) = \beta$ then $\mathrm{RS}(\mathcal{T}_\varphi) = \alpha$ for some (α-ranking) sentence φ. Moreover, there are $\mathrm{ds}(\mathcal{T})$ pairwise $\mathcal{T}$-inconsistent β-ranking sentences for $\mathcal{T}$, and if $\alpha < \beta$ then there are infinitely many pairwise $\mathcal{T}$-inconsistent α-ranking sentences for $\mathcal{T}$.

Theorem 5

Let $\mathcal{T}$ be a family of a countable language Σ and with $\mathrm{RS}(\mathcal{T}) = \infty$, α be a countable ordinal, $n \in \omega \setminus \{0\}$. Then there is a d_∞-definable subfamily $\mathcal{T}^* \subset \mathcal{T}$ such that $\mathrm{RS}(\mathcal{T}^*) = \alpha$ and $\mathrm{ds}(\mathcal{T}^*) = n$.

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[9] Markhabatov N. D., Sudoplatov S. V. Definable subfamilies of theories, related calculi and ranks // Siberian Electronic Mathematical Reports. — 2020. — Vol. 17. — P. 700–714.

Separability of closures[10]

Theorem 6

For any two disjoint subfamilies $\mathcal{T}_1$ and $\mathcal{T}_2$ of an E-closed family $\mathcal{T}$ the following conditions are equivalent:
(1) $\mathcal{T}_1$ and $\mathcal{T}_2$ are separated by some sentence φ: $\mathcal{T}_1 \subseteq \mathcal{T}_\varphi$ and $\mathcal{T}_2 \subseteq \mathcal{T}_{\neg\varphi}$;
(2) E-closures of $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint in $\mathcal{T}$: $\mathrm{Cl}_E(\mathcal{T}_1) \cap \mathrm{Cl}_E(\mathcal{T}_2) \cap \mathcal{T} = \emptyset$;
(3) E-closures of $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint: $\mathrm{Cl}_E(\mathcal{T}_1) \cap \mathrm{Cl}_E(\mathcal{T}_2) = \emptyset$.

[10] Sudoplatov S. V. Hierarchy of families of theories and their rank characteristics // Bulletin of Irkutsk State University. Series Mathematics. — 2020. — Vol. 33. — P. 80–95.

Generating sets[11]

Definition. Let $\mathcal{T}_0$ be a family of theories. A subset $\mathcal{T}_0' \subseteq \mathcal{T}_0$ is said to be *generating* if $\mathcal{T}_0 = \mathrm{Cl}_E(\mathcal{T}_0')$. The generating set $\mathcal{T}_0'$ (for $\mathcal{T}_0$) is *minimal* if $\mathcal{T}_0'$ does not contain proper generating subsets. A minimal generating set $\mathcal{T}_0'$ is *least* if $\mathcal{T}_0'$ is contained in each generating set for $\mathcal{T}_0$.

[11] Sudoplatov S. V. Closures and generating sets related to combinations of structures // Bulletin of Irkutsk State University. Series Mathematics. — 2016. — Vol. 16. — P. 131–144.

Generating sets[12]

Theorem 7

If $\mathcal{T}_0'$ is a generating set for a E-closed set $\mathcal{T}_0$ then the following conditions are equivalent:
(1) $\mathcal{T}_0'$ is the least generating set for $\mathcal{T}_0$;
(2) $\mathcal{T}_0'$ is a minimal generating set for $\mathcal{T}_0$;
(3) any theory in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0')_\varphi$, i.e., for any $T \in \mathcal{T}_0'$ there is $\varphi \in T$ such that $(\mathcal{T}_0')_\varphi = \{T\}$;
(4) any theory in $\mathcal{T}_0'$ is isolated by some set $(\mathcal{T}_0)_\varphi$, i.e., for any $T \in \mathcal{T}_0'$ there is $\varphi \in T$ such that $(\mathcal{T}_0)_\varphi = \{T\}$.

[12] Sudoplatov S. V. Closures and generating sets related to combinations of structures // Bulletin of Irkutsk State University. Series Mathematics. — 2016. — Vol. 16. — P. 131–144.

Definition. Let Σ be a language, $\varphi \rightleftharpoons \varphi(\overline{x})$ be a formula in $F(\Sigma)$, P_s be a subclass of the class $K(\Sigma)$ of all structures $\mathcal{A}$ in the language Σ. We say that $\varphi(\overline{x})$ *partially* (respectively, *totally*) *satisfies* P_s, denoted by $\varphi \rhd_{\rm ps} P_s$ or $\varphi \rhd^{\exists}_s P_s$ ($\varphi \rhd_{\rm ts} P_s$ or $\varphi \rhd^{\forall}_s P_s$), if there are $\mathcal{A} \in P_s$ and $\overline{a} \in A$ (for any $\mathcal{A} \in P_s$ there is $\overline{a} \in A$) such that $\mathcal{A} \models \varphi(\overline{a})$.

If $P_{\rm is}$ is a subclass of the class ${\rm ITK}(\Sigma)$ of isomorphism types for the class $K(\Sigma)$ then we say that $\varphi(\overline{x})$ *partially* (respectively, *totally*) *satisfies* $P_{\rm its}$, denoted by $\varphi \rhd_{\rm pits} P_{\rm its}$ or $\varphi \rhd^{\exists}_{\rm its} P_{\rm its}$ ($\varphi \rhd_{\rm tits} P_{\rm its}$ or $\varphi \rhd^{\forall}_{\rm its} P_{\rm its}$) if $\varphi \rhd_{\rm ps} P_s$ ($\varphi \rhd_{\rm ts} P_s$, where P_s consists of all structures whose isomorphism types belong to $P_{\rm its}$. If P_t is a subset of the set $\mathcal{T}_\Sigma$ of all complete theories in the language Σ then we say that $\varphi(\overline{x})$ *partially* (respectively, *totally*) *satisfies* P_t, denoted by $\varphi \rhd_{\rm pt} P_t$ or $\varphi \rhd^{\exists}_t P_t$ ($\varphi \rhd_{\rm tt} P_t$ or $\varphi \rhd^{\forall}_t P_t$), if there are $T \in P_t$, $\mathcal{M} \models T$, and $\overline{a} \in M$ (for any $T \in P_t$ there are $\mathcal{M} \models T$ and $\overline{a} \in M$) such that $\mathcal{M} \models \varphi(\overline{a})$.

For a property P_s we denote by $\mathrm{ITK}(P_s)$ the class of isomorphism types for structures in P_s, and by $\mathrm{Th}(P_s)$ the set $\{T \in \mathcal{T}_\Sigma \mid \mathcal{A} \models T \text{ for some } \mathcal{A} \in P_s\}$. For a property P_{its} we denote by $K(P_{\mathrm{its}})$ the class of all structures whose isomorphism types are represented in P_{its}, and by $\mathrm{Th}(P_{\mathrm{its}})$ the set $\mathrm{Th}(K(P_{\mathrm{its}}))$. For a property P_t we denote by $K(P_t)$ the class of all models of theories in P_t, and by $\mathrm{ITK}(P_t)$ the class $\mathrm{ITK}(K(P_t))$.

Proposition 6

For any formula $\varphi \in F(\Sigma)$ and properties P_s, $P_{\rm its}$, P_t the following conditions hold:
(1) $\varphi \rhd_{\rm ps} P_s$ iff $\varphi \rhd_{\rm pits} {\rm ITK}(P_s)$, and iff $\varphi \rhd_{\rm pt} {\rm Th}(P_s)$;
(2) $\varphi \rhd_{\rm ts} P_s$ iff $\varphi \rhd_{\rm tits} {\rm ITK}(P_s)$, and iff $\varphi \rhd_{\rm tt} {\rm Th}(P_s)$;
(3) $\varphi \rhd_{\rm pits} P_{\rm its}$ iff $\varphi \rhd_{\rm ps} K(P_{\rm its})$, and iff $\varphi \rhd_{\rm pt} {\rm Th}(P_{\rm its})$;
(4) $\varphi \rhd_{\rm tits} P_{\rm its}$ iff $\varphi \rhd_{\rm ts} K(P_{\rm its})$, and iff $\varphi \rhd_{\rm tt} {\rm Th}(P_{\rm its})$;
(5) $\varphi \rhd_{\rm pt} P_t$ iff $\varphi \rhd_{\rm ps} K(P_t)$, and iff $\varphi \rhd_{\rm pits} {\rm ITK}(P_t)$;
(6) $\varphi \rhd_{\rm tt} P_t$ iff $\varphi \rhd_{\rm ts} K(P_t)$, and iff $\varphi \rhd_{\rm tits} {\rm ITK}(P_t)$.

In the items (3) and (4) the class $K(P_{\rm its})$ can be replaced by a subclass K' such that $\mathrm{ITK}(K') = P_{\rm its}$. Similarly, in the items (5) and (6) the class $K(P_t)$ can be replaced by a subclass K' such that $\mathrm{Th}(K') = P_t$, and independently $\mathrm{ITK}(P_t)$ can be replaced by a subclass K'' such that $\mathrm{Th}(K'') = P_t$.

By Proposition 6 semantic properties P_s and $P_{\rm its}$ can be naturally transformed into syntactic ones P_t, and vice versa. It means that natural model-theoretic properties such as ω-categoricity, stability, simplicity etc. can be formulated both for theories, for structures and for their isomorphism types. The links between $\rhd$-relations which pointed out in Proposition 6 allow to reduce our consideration to the relations $\rhd_{\rm pt}$ and $\rhd_{\rm tt}$. Besides, for the simplicity we will principally consider sentences φ instead of formulas in general. Reductions of formulas $\psi(\overline{x})$ to sentences use the operators $\psi(\overline{x}) \mapsto \forall \overline{x}\, \psi(\overline{x})$ and $\psi(\overline{x}) \mapsto \exists \overline{x}\, \psi(\overline{x})$.

Theorem 8

For any sentence $\varphi \in \mathrm{Sent}(\Sigma)$ and a property $P_t \subseteq \mathcal{T}_\Sigma$ the following conditions are equivalent:
(1) $\varphi \rhd_{\mathrm{pt}} P_t$,
(2) $\varphi \rhd_{\mathrm{pt}} \mathrm{Cl}_E(P_t)$,
(3) $\varphi \rhd_{\mathrm{pt}} P'_t$ for any/some P'_t with $\mathrm{Cl}_E(P'_t) = \mathrm{Cl}_E(P_t)$.

Theorem 9

For any sentence $\varphi \in \mathrm{Sent}(\Sigma)$ and a property $P_t \subseteq \mathcal{T}_\Sigma$ the following conditions are equivalent:
(1) $\varphi \rhd_{\mathrm{tt}} P_t$,
(2) $\varphi \rhd_{\mathrm{tt}} \mathrm{Cl}_E(P_t)$,
(3) $\varphi \rhd_{\mathrm{tt}} P'_t$ for any/some P'_t with $\mathrm{Cl}_E(P'_t) = \mathrm{Cl}_E(P_t)$.

Corollaries

Corollary 1

For any properties $P_1, P_2 \subseteq \mathcal{T}_\Sigma$ the following conditions hold:
(1) there exists $\varphi \in \mathrm{Sent}(\Sigma)$ such that $\varphi \rhd_{\rm pt} P_1$ and $\neg\varphi \rhd_{\rm pt} P_2$ iff P_1 and P_2 are nonempty and $|P_1 \cup P_2| \geq 2$; in particular, there exists $\varphi \in \mathrm{Sent}(\Sigma)$ such that $\varphi \rhd_{\rm pt} P_1$ and $\neg\varphi \rhd_{\rm pt} P_1$ iff $|P_1| \geq 2$;
(2) there exists $\varphi \in \mathrm{Sent}(\Sigma)$ such that $\varphi \rhd_{\rm tt} P_1$ and $\neg\varphi \rhd_{\rm tt} P_2$ iff $\mathrm{Cl}_E(P_1) \cap \mathrm{Cl}_E(P_2) = \emptyset$.

Corollary 2

For any nonempty property $P_t \subseteq \mathcal{T}_\Sigma$ the following conditions hold:
(1) the set $\bigcap P_t$ forms a filter $\bigcap P_t/\equiv$ on $\{\equiv(\varphi) \mid \varphi \in \mathrm{Sent}(\Sigma)\}$ with respect to $\vdash$;
(2) the filter $\bigcap P_t/\equiv$ is principal iff $\bigcap P_t$ is forced by some its sentence, i.e., $\bigcap P_t$ is a finitely axiomatizable theory, which is incomplete for $|P_t| \geq 2$;
(3) the filter $\bigcap P_t/\equiv$ is an ultrafilter iff P_t is a singleton.

Definition. For a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ and a property $P = P_t \subseteq \mathcal{T}_\Sigma$ we put $\mathrm{RS}_P(\varphi) = \mathrm{RS}(P_\varphi)$, and $\mathrm{ds}_P(\varphi) = \mathrm{ds}(P_\varphi)$ if $\mathrm{ds}(P_\varphi)$ is defined.
If $P = \mathcal{T}_\Sigma$ then we omit P and write $\mathrm{RS}(\varphi)$, $\mathrm{ds}(\varphi)$ instead of $\mathrm{RS}_P(\varphi)$ and $\mathrm{ds}_P(\varphi)$, respectively.

Definition. For a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ and a property $P \subseteq \mathcal{T}_\Sigma$ we say that φ is *P-totally transcendental* if $\mathrm{RS}_P(\varphi)$ is an ordinal.
A sentence φ is *co-(P)-totally transcendental* if $\neg\varphi$ is P-totally transcendental.
We omit P and say about totally transcendental and co-totally transcendental sentences if $P = \mathcal{T}_\Sigma$.

Theorem 10

For a language Σ there is a totally transcendental sentence $\varphi \in \mathrm{Sent}(\Sigma)$ iff Σ has finitely many predicate symbols.

Ranks and spectra for sentences and properties

Definition. For a language Σ, a property $P \subseteq \mathcal{T}_\Sigma$, an ordinal α and a natural number $n \geq 1$, a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ is called (P, α, n)-*(co-)rich* if $\mathrm{RS}_P(\varphi) = \alpha$ and $\mathrm{ds}_P(\varphi) = n$ (respectively, $\mathrm{RS}_P(\neg\varphi) = \alpha$ and $\mathrm{ds}_P(\neg\varphi) = n$).
A sentence $\varphi \in \mathrm{Sent}(\Sigma)$ is called (P, ∞)-*(co-)rich* if $\mathrm{RS}_P(\varphi) = \infty$ (respectively, $\mathrm{RS}_P(\neg\varphi) = \infty$).
If $P = \mathcal{T}_\Sigma$ we write that φ is (α, n)-(co-)rich instead of (P, α, n)-(co-)rich, and ∞-(co-)rich instead of (P, ∞)-(co-)rich.
If for a property P there is a $(P, *)$-(co-)rich sentence φ, we say that P has a $(P, *)$-*(co-)rich sentence*, where $* = \alpha, n$ or $\alpha = \infty$.

Theorem 11

(1) If a property $P \subseteq \mathcal{T}_\Sigma$ has a (P, α, m)-rich sentence φ which is (P, β, n)-co-rich then $\mathrm{RS}(P) = \max\{\alpha, \beta\}$, $\mathrm{ds}(P) = m$ for $\alpha > \beta$, $\mathrm{ds}(P) = n$ for $\alpha < \beta$, and $\mathrm{ds}(P) = m + n$ for $\alpha = \beta$.
(2) If for a property $P \subseteq \mathcal{T}_\Sigma$, $\mathrm{RS}(P) = \alpha$ and $\mathrm{ds}(P) = n$, then for each sentence $\varphi \in \mathrm{Sent}(\Sigma)$ the following assertions hold:
(i) $\mathrm{RS}_P(\varphi) \leq \alpha$,
(ii) if $\mathrm{RS}_P(\varphi) = \alpha$ then φ is (P, α, m)-rich for some $m \leq n$, and for $m = n$ either $\varphi \rhd_{\mathrm{tt}} P$ or φ is (P, β, k)-co-rich for some $\beta < \alpha$ and $k \in \omega$, and if $m < n$ then φ is $(P, \alpha, n - m)$-co-rich.

Ranks and spectra for sentences and properties

By Theorem 11 for any e-totally transcendental property P and any $\alpha \leq \mathrm{RS}(P)$ there are s-definable subfamilies P_φ with $\mathrm{RS}(P_\varphi) = \alpha$. Similarly all values $m \leq \mathrm{ds}(P)$ are also realized by appropriate s-definable subfamilies.

Thus the *spectrum* $\mathrm{Sp}_{\mathrm{Rd}}(P)$ for the pairs $(\mathrm{RS}_P(\varphi), \mathrm{ds}_P(\varphi))$ with nonempty P_φ forms the set

$$\{(\mathrm{RS}(P), m) \mid 1 \leq m \leq \mathrm{ds}(P)\} \cup \{(\alpha, m) \mid \alpha < \mathrm{RS}(P), m \in \omega \setminus \{0\}\},$$

which is an initial segment $O[(\beta, n)]$ consisting of all pairs $(\alpha, m) \in \mathrm{Ord} \times (\omega \setminus 0)$ with $\alpha \leq \beta$ and $m \leq n$ for $\alpha = \beta$, $\mathrm{RS}(P) = \beta$, $\mathrm{ds}(P) = n$.

Theorem 12

For any nonempty property $P \subseteq \mathcal{T}_\Sigma$ one of the following possibilities holds for some $\beta \in \mathrm{Ord}$ and $n \in \omega \setminus \{0\}$:
(1) $\mathrm{Sp}_{\mathrm{Rd}}(P) = O[(\beta, n)]$,
(2) $\mathrm{Sp}_{\mathrm{Rd}}(P) = \{\infty\}$,
(3) $\mathrm{Sp}_{\mathrm{Rd}}(P) = O[(\beta, n)] \cup \{\infty\}$.
All possibilities above are realized by appropriate languages Σ and properties $P \subseteq \mathcal{T}_\Sigma$.

Ranks and spectra for sentences and properties

The following theorem is shown in: Markhabatov N. D., Sudoplatov S. V. Ranks for families of all theories of given languages // Eurasian Mathematical Journal. — 2021. — Vol. 12, No. 2.

Theorem 13

Let $\mathcal{T}$ be a family of a countable language Σ and with $\mathrm{RS}(\mathcal{T}) = \infty$, α be a countable ordinal, $n \in \omega \setminus \{0\}$. Then there is a d_∞-definable subfamily $\mathcal{T}^* \subset \mathcal{T}$ such that $\mathrm{RS}(\mathcal{T}^*) = \alpha$ and $\mathrm{ds}(\mathcal{T}^*) = n$.

The latter two Theorems immediately imply:

Corollary 3

Let $\mathcal{T}$ be a family of a countable language Σ and with $\mathrm{RS}(\mathcal{T}) = \infty$, α be a countable ordinal, $n \in \omega \setminus \{0\}$. Then there is a d_∞-definable property $P \subset \mathcal{T}$ such that $\mathrm{Sp}_{\mathrm{Rd}}(P) = O[(\alpha, n)]$.

Links between sentences and properties

For a cardinality $\lambda\geq 1$, a sentence $\varphi\in\mathrm{Sent}(\Sigma)$ and a property $P\subseteq\mathcal{T}_\Sigma$ we write $\varphi\rhd^{\lambda}_{\rm pt}P$ if φ satisfies exactly λ theories in P, i.e., $|P_\varphi|=\lambda$.

By the definition if $P\neq\emptyset$ and $\varphi\rhd_{\rm tt}P$ then $\varphi\rhd^{|P|}_{\rm pt}P$, and conversely $\varphi\rhd^{|P|}_{\rm pt}P$ implies $\varphi\rhd_{\rm tt}P$ for finite P. For infinite P the converse implication can fail. Moreover, since infinite sets can be divided into two parts of same cardinality, one can easily introduce an expansion P' of P by a 0-ary predicate Q such that $Q\rhd^{|P'|}_{\rm pt}P'$ and $\neg Q\rhd^{|P'|}_{\rm pt}P'$, implying that $Q\not\rhd_{\rm tt}P'$.

Spectra for properties

For a property P we denote by $\mathrm{Sp}_{\mathrm{pt}}(P)$ the set $\{\lambda \mid \varphi \rhd^{\lambda}_{\mathrm{pt}} P$ for some sentence $\varphi\}$. This set is called the pt-*spectrum* of P.

Theorem 14

For any nonempty property $P \subseteq \mathcal{T}_\Sigma$ one of the following conditions holds:
(1) $\mathrm{Sp}_{\mathrm{pt}}(P) = (n+1) \setminus \{0\}$ for some $n \in \omega \setminus \{0\}$; it is satisfied iff P is finite with $|P| = n$;
(2) $\mathrm{Sp}_{\mathrm{pt}}(P) = Y \cup (n+1) \setminus \{0\}$ for some nonempty set $Y \subseteq |P|$ of infinite cardinalities and $n \in \omega \setminus \{0\}$;
(3) $\mathrm{Sp}_{\mathrm{pt}}(P) = Y \cup \omega \setminus \{0\}$ for some nonempty set $Y \subseteq |P|$ of infinite cardinalities;
(4) $\mathrm{Sp}_{\mathrm{pt}}(P) = Y$ for some nonempty set $Y \subseteq |P|$ of infinite cardinalities.
All values $(n+1) \setminus \{0\}$, $Y \cup (n+1) \setminus \{0\}$, $Y \cup \omega \setminus \{0\}$, and Y, for a nonempty set Y of infinite cardinalities and $n \in \omega \setminus \{0\}$, are realized as $\mathrm{Sp}_{\mathrm{pt}}(P)$ for an appropriate property P.

Spectra for properties

Theorem 15

For any nonempty E-closed property $P \subseteq \mathcal{T}_\Sigma$ with at most countable language Σ one of the following possibilities holds:
(1) $\mathrm{Sp}_{\mathrm{pt}}(P) = (n+1) \setminus \{0\}$ for some $n \in \omega \setminus \{0\}$, if P is finite with $|P| = n$;
(2) $\mathrm{Sp}_{\mathrm{pt}}(P) = \{2^\omega\} \cup (n+1) \setminus \{0\}$ for some $n \in \omega$, if P is infinite and has n isolated points;
(3) $\mathrm{Sp}_{\mathrm{pt}}(P) = (\omega+1) \setminus \{0\}$, if P is infinite and totally transcendental;
(4) $\mathrm{Sp}_{\mathrm{pt}}(P) = \{\omega, 2^\omega\} \cup \omega \setminus \{0\}$, if P has an infinite totally transcendental definable subfamily but P itself is not totally transcendental;
(5) $\mathrm{Sp}_{\mathrm{pt}}(P) = \{2^\omega\} \cup \omega \setminus \{0\}$, if P has infinitely many isolated points but does not have infinite totally transcendental definable subfamilies.

Definition. (Cf. [13]) For a property $P \subseteq \mathcal{T}_\Sigma$ a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ is called *P-generic* if $\mathrm{RS}_P(\varphi) = \mathrm{RS}(P)$, and $\mathrm{ds}_P(\varphi) = \mathrm{ds}(P)$ if $\mathrm{ds}(P)$ is defined.
If $P = \mathcal{T}_\Sigma$ then we omit P and a P-generic sentence is called *generic*.

[13] Poizat, B. Groupes Stables. Nur Al-Mantiq Wal-Ma'rifah: Villeurbanne, France 1987. Truss, J.K. Generic Automorphisms of Homogeneous Structures // Proceedings of the London Mathematical Society. 1992, 65:3, 121–141. Tent, K., Ziegler, M. A Course in Model Theory // Lecture Notes in Logic. No. 40. Cambridge University Press: Cambridge, UK, 2012.

Generic sentences

Proposition 7

Any P-generic sentence φ is $(P, \mathrm{RS}(P), \mathrm{ds}(P))$-rich if $\mathrm{RS}(P)$ is an ordinal, and (P, ∞)-rich if $\mathrm{RS}(P) = \infty$. And vice versa, each $(P, \mathrm{RS}(P), \mathrm{ds}(P))$-rich sentence, for an ordinal $\mathrm{RS}(P)$, is P-generic, and each (P, ∞)-rich sentence, for $\mathrm{RS}(P) = \infty$, is P-generic.

Corollary 4

If a property $P \subseteq \mathcal{T}_\Sigma$ is finite and $\varphi \in \mathrm{Sent}(\Sigma)$ then $\varphi \rhd_{\mathrm{tt}} P$ iff φ is P-generic.

Proposition 8

For a property $P \subseteq \mathcal{T}_\Sigma$ there is a P-generic sentence $\varphi \in \mathrm{Sent}(\Sigma)$ with minimal/least P_φ iff P is finite. If that φ exists then $P_\varphi = P$.

Corollary 5

For any property $P \subseteq \mathcal{T}_\Sigma$ with $\mathrm{RS}(P) = \alpha \in \mathrm{Ord}$ and any sentence $\varphi \in \mathrm{Sent}(\Sigma)$ either φ is P-generic or $\neg\varphi$ is P-generic, or, for $\mathrm{ds}(P) > 1$ with non-P-generic φ and $\neg\varphi$, φ is represented as a disjunction of k $(P, \alpha, 1)$-rich sentences and $\neg\varphi$ is represented as a disjunction of m $(P, \alpha, 1)$-rich sentences such that $k + m = \mathrm{ds}(P)$, $k > 0$, $m > 0$.

Theorem 16

(1) For any nonempty property $P \subseteq \mathcal{T}_\Sigma$ there are $\mathrm{ds}(P)$ P-generic theories if P is totally transcendental, and at least continuum many if P is not totally transcendental. In the latter case either all theories in P are P-generic if $\mathrm{Sp}_{\mathrm{Rd}}(P) = \{\infty\}$, or P has at least $\beta \cdot \omega + n$ non-P-generic theories if $\mathrm{Sp}_{\mathrm{Rd}}(P) = O[(\beta, n)] \cup \{\infty\}$.
(2) The CB-rank of each P-generic theory equals $\mathrm{RS}(P)$.

Generic sentences

Definition. For a property $P \subseteq \mathcal{T}_\Sigma$ a sentence $\varphi \in \mathrm{Sent}(\Sigma)$ is called *P-complete* if φ isolates a unique theory T in P, i.e., P_φ is a singleton. In such a case the theory $T \in P_\varphi$ is called *P-finitely axiomatizable* (by the sentence φ).

Proposition 9

For any nonempty property $P \subseteq \mathcal{T}_\Sigma$ a P-finitely axiomatizable theory T is P-generic iff P is finite.

The obtained results are published in: Sudoplatov S.V. Formulas and Properties, Their Links and Characteristics // Mathematics. 2021, Vol. 9, Issue 12. 1391.