DEGREES OF REPRESENTABILITY OF LINEAR ORDERS

N.Kh. Kasymov, R.N. Dadazhanov, S.K. Zhavliev

National University of Uzbekistan named after M. Ulugbek

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Preliminary Information

Following Yu.L. Ershov. [1], Yu.L. Ershov [2], S.S. Goncharov, Yu.L. Ershov [3], give the basic definitions.

Definition 1.1.

Algorithmic representation of a countable system $\mathfrak{M} = \langle M; \Sigma \rangle$ *of effective signature* Σ *is any such mapping of* μ *of the set of natural numbers* ω *to the main set of* M *of system* $\mathfrak{M}$*, for which there is an effective family* F_Σ *computable functions representing* Σ*-operations of the system* $\mathfrak{M}$ *in the numbering of* μ*, that is, any* $\sigma \in \Sigma$*-operation represented by its corresponding computable function* $f_\sigma \in F_\Sigma$*, that* $\forall \overline{x}\ (\sigma\mu(\overline{x}) = \mu f_\sigma\ (\overline{x}))$.

[1] Yu.L. Ershov. Theory of numberings. Moscow., Nauka, 1977. (in Russian)

[2] Yu.L. Ershov. Solvability Problems and Constructive Models. Moskow., Nauka, 1980. (in Russian)

[3] S.S. Goncharov, Yu.L. Ershov. Constructive Models. Siberian School of Algebra and Logic, Kluwer Academic / Plenum Publishers, New York ets., 2000.

Diagram 1

$$\begin{tikzcd}
\mathfrak{M} \arrow[r, "\sigma \in F_\Sigma"] & \mathfrak{M} \\
\omega \arrow[u, "\nu"] \arrow[r, "f_\sigma"] & \omega \arrow[u, "\nu"']
\end{tikzcd}$$

Definition 1.2.

If μ is algorithmic representation of system $\mathfrak{M}$, then the pair $(\mathfrak{M}, \mu)$ is called a numbered system.

All the homomorphisms of numbered systems we consider are computable, that is, they are supported by functions that are computable on representations in the following sense.

Definition 1.3.

The homomorphism $\varphi : \mathfrak{M} \to \mathfrak{N}$ is called the computable homomorphism (or morphism) of the numbered models $(\mathfrak{M}, \mu) \to (\mathfrak{N}, \nu)$, if there is such a computable function h, that $\varphi\mu = \nu h$.

Further, by the homomorphisms of numbered systems, we mean their morphisms, i.e. we work in the category of numbered systems with morphisms as effective on numbers of homomorphisms.

Diagram 2

$$
\begin{tikzcd}
\mathfrak{M} \arrow[r, "\varphi"] & \mathfrak{N} \\
\omega \arrow[u, "\mu"] \arrow[r, "h"] & \omega \arrow[u, "\nu"']
\end{tikzcd}
$$

The kernel of the representation μ of system $\mathfrak{M}$ is the equivalence $\{\langle x, y\rangle | \mu x = \mu y\}$. If μ is a representation, then its kernel will be denoted through $ker(\mu)$.

Let η be a fixed equivalence on ω and $\mathfrak{M}$ be a system with a representation with a kernel equal to η. Then the system $\mathfrak{M}$ will be called represented over η (or η-system).

With a fixed system, classical is the problem of studying its various representations and the relations between them, in particular, the problem of the existence of good representations (for example, computable) and the relations between them (including uniqueness, accurate to computable isomorphism, representation).

On the other hand, you can fix the representation kernel and study the general properties of systems that have representations with this kernel. This approach seems appropriate from the point of view of the theory of representations of systems in the framework of theoretical computer science.

Definition 1.4.

A numbered system, the kernel and all the main relations of which are computable (enumerable, co-enumerable) is called computable (positive, negative).

If $(\mathfrak{M}, \mu), (\mathfrak{N}, \nu)$ are two numbered systems, then we say that $(\mathfrak{M}, \mu)$ is reduced to $(\mathfrak{N}, \nu)$ if there is a computable isomorphism from $\mathfrak{M}$ onto $\mathfrak{N}$. Let's say that η_0 m is reduced to η_1 (in the notation $\eta_0 \leqslant_m \eta_1$) if there is such a computable function g that $x = y \pmod{\eta_0} \Leftrightarrow g(x) = g(y) \pmod{\eta_1}$ и $\forall y \exists x (g(x) = y \pmod{\eta_1})$. If $\eta_0 \leqslant_m \eta_1 \wedge \eta_1 \leqslant_m \eta_0$, we will assume that $\eta_0 \equiv_m \eta_1$. Then $\equiv_m$ is an equivalence on the classes of which the induced $\leqslant_m$ partial order is correctly defined, which we will denote with the same sign. As usual, m-degree of the equivalence η (in the notation $d_m(\eta)$) we call the set $\{\eta' | \eta \equiv_m \eta'\}$.

Three theses on the priority of negativity over positivity

Thesis 1

Algorithmical defined topological neighborhoods, the presence of which allows to solve the key problem of recognition in algorithmically defined complex systems. For negative (and, more generally, for computably separable) equivalences, the topological spaces generated by computable subsets will be T_4-spaces.

Thesis 2

Representability over equivalences.

Thesis 3

Structural theory of computably separable numbered systems – a numbered system is computably separable if and only if it is approximated by negative systems.

Known [4], that there is a positively representable linear order that has no computable representations. On the other hand ,[5], every negatively represented linear order has a computable representation. Therefore, the question of the existence of computable representations for negatively representable linear orders with endomorphisms is fundamental.

Theorem 2.1.

There is a negatively representable linear order with two endomorphisms, which does not have positive representations.

Corollary 2.1.

There is a negatively representable linear order with endomorphisms that does not have solvable representations.

[4] L. Feiner. Hierarchies of Boolean algebras. Journal of Symbolic Logic, **35** (2), 363–373, 1970.

[5] H.Kh. Kasymov, R.N. Dadazhanov. Negative dense linear orders. Sib. matem. journal, **58** (6), 1306-1331, 2017.

A classical example of order and endomorphism on it is a natural series with a natural order and function $x+1$.
The algebra $S = \langle \omega; s \rangle$ (without an order relation) is computably stable with respect to positive representations, i.e. any of its positive representations is computably isomorphic to the simplest.
On the other hand ,[6], there is an nonsolvable negative representation of this algebra. Against this background, the importance of order from an algorithmic point of view demonstrates the following

Proposition 2.1.

Any negative representation of the natural order $S_{\leq}$ of natural numbers with the function $x+1$ is solvable.

[6] B. Khoussainov, T. Slaman, P. Semukhin. $\prod_1^0$-Presentasions of Algebras. Archive for Mathematical Logic ,**45** (6), 769-781, 2006.

Proposition 3.1.

If the linear order is positively (negatively) represented over the negative (positive) equivalence, then both the order and equivalence are algorithmic solvable.

Further, taking into account proposition 3.1, expression "it is negatively (positively) representable over negative (positive) equivalence" often we will reduce to "it is representable over negative (positive) equivalence", believing by default that the negative (positive) representability of an order over negative (positive) equivalence is meant.
We will be primarily interested in the representability of orders over negative equivalents. A brief overview of the results on the representability of orders over positive equivalences will be given below.

For negative equivalence η, we denote through $L(\eta)$ the class of all linear orders negatively represented over η, i.e. the types of isomorphisms of such structures and on the set Π of all negative equivalences on ω we define the following binary relation $\leqslant_{ln}$:

$$\eta_1 \leqslant_{ln} \eta_2 \Leftrightarrow L(\eta_1) \subseteq L(\eta_2),$$

which is a preorder on the set Π and its symmetric close $\equiv_{ln-e}$ is an equivalence by which factorization breaks the set of all negative equivalences into classes $\equiv_{ln}$-equivalence.
The partial order $\langle \Pi / \equiv_{ln}; \leqslant_{ln} \rangle$ will be called *structure of negative representability of linear orders*, and its elements – *degrees of negative representability of linear orders*. Further, if it is clear what is involved, the structure of negative representability of linear orders will be called simply the structure of negative representability, and its elements – degrees.

To shorten symbols through $d_{ln}(\eta)$, we will denote the degree of negative representation of equivalence η. Let us also say that a linear order is represented over a given degree if it is represented over some (and therefore over any) equivalence from this degree.
Nonformally, $\equiv_{ln}$ -equivalence of two negative equivalences means the coincidence of the types of isomorphisms of linear orders represented above them.
The structure of negative representability of linear orders reflects the algorithmic nature of equivalences in terms of the possibilities they provide for the realization of important objects over them, to which linear orders certainly belong. Clearly, the higher the $\leqslant_{ln}$ is the $\equiv_{ln}$-degree, the more realizations it provides. However, a priori, it cannot be claimed that $\leqslant_{ln}$-top $\equiv_{ln}$-degrees are always preferable to $\leqslant_{ln}$-lower degrees.

So, for example, over any negative equivalence, we present a completely meaningful class of linear orders, and if the task is to determine the maximum "refined" class of η-like (η – the type of ordering of rational numbers) orders, then it may turn out that it is more expedient to choose implementations in the lower $\equiv_{ln}$-degrees. Such an approach may also be useful in theoretical computer science.

Note that finite negative equivalences generate isolated degrees in the structure of negative representability. From a descriptive point of view, it is reasonable to consider orders over infinite equivalences. It is in the assumption of the absence of finite degrees that we will conduct our consideration. Discarding all $\equiv_{ln}$-classes containing finite equivalences, we get the constraints of the relations $\leqslant_{ln}, \equiv_{ln}$ on infinite negative equivalences. Everywhere, the structure of negative representability is considered in the context of the absence of degrees containing finite equivalences.

We say that the linear order $\langle L; \preceq, \varepsilon_0, \varepsilon_1 \ldots \rangle$ with endomorphisms $\varepsilon_0, \varepsilon_1 \ldots$ is computably (positively, negatively) *representable over the equivalence* η on the set of natural numbers ω, if there is such a numbering of ν for L with a numbering equivalence of η, in which all endomorphisms are computable, and the sets of ν-numbers of equality and order relations are computable (positive, accordingly negative).
For negative equivalence η through $L_e(\eta)$, we denote the class of all linear orders with endomorphisms negatively represented over η and on set Π we enter the following binary relation $\leqslant_{ln-e}$:

$$\eta_1 \leqslant_{ln-e} \eta_2 \Leftrightarrow L_e(\eta_1) \subseteq L_e(\eta_2),$$

which is a preorder on the set Π and its symmetric closure $\equiv_{ln-e}$ is an equivalence by which factorization breaks the set of all negative equivalences into $\equiv_{ln-e}$-equivalence classes.

The partially ordered set $\langle \Pi / \equiv_{ln-e}; \leqslant_{ln-e} \rangle$ will be called *structure of negative representability of linear orders with endomorphisms*, and its elements – *degrees of negative representability of linear orders with endomorphisms*.

Let Σ be the set of infinite positive equivalences and the relation $\eta_1 \leqslant_{lp} \eta_2$ on Σ means that every linear order, positively represented over η_1, positively represented over η_2. Similarly to the negative case, by symmetrically closing the preorder $\leqslant_{lp}$ and factorizing with respect to the obtained equivalence relation on the set of all infinite positive equivalences, we obtain the structure of positive representability of linear orders $\langle \Sigma / \equiv_{lp}; \leqslant_{lp} \rangle$, which turned out to be completely different, than the structure of negative representability of linear orders.

Finally, we define the relation $\eta_1 \leqslant_{lp-e} \eta_2$ on a set of positive equivalences, which means that every linear order with endomorphisms, positively represented over η_1, is positively represented over η_2.

Similarly, the structure of positive representability of linear orders with endomorphisms $\langle \Sigma / \equiv_{lp-e}; \leqslant_{lp-e} \rangle$ is determined.

Proposition 3.2.

Let η_e be a perfect positive equivalence with a compressed characteristic transversal. Then the degree $d_{lp}(\eta_e)$ is the smallest element in the structure of lp-degrees.

Thus, a non-empty set of all those positive equivalences over which we will not represent any linear order at all form one *lp*-degre of positive representability and this degree is $d_{lp}(\eta_{ersh})$, which defines an empty class of linear orders represented above it. Obviously, this degree is the smallest element in the structure of positive representability of linear orders, as noted in[7], although the indicated work did not use the equivalence η_e.

[7] E. Fokina, B. Khoussainov, P. Semukhin, D. Turetskiy. Linear Orders Realized by C.E. Equivalence Relations. Journal of Symbolic Logic, **81** (2), 463–482, 2016.

In the same work, in particular, it is shown that the structure $\langle \Sigma / \equiv_{lp}; \leqslant_{lp} \rangle$ does not have the largest element, but has the maximum (it will be the degree $d_{lp}(id\ \omega)$), there is an infinitely decreasing chain of degrees of positive representation and there are incomparite degrees (an analogue of the Friedberg-Muchnik theorem for degrees of positive representability of linear orders).

In the framework of the ideology of theoretical computer science, instead of linear orders, other objects can also be considered, including universal algebras (without fixing a signature), which are widely used in abstract data types and object-oriented programming. At the same time, reasonable extensions of the class of equivalences considered also allow you to obtain structures with meaningful properties. However, it seems that it is low degrees that can be of significant interest from the point of view of theoretical computer science.

Corollary 3.1.

There are incomparable degrees of negative representability of linear orders with endomorphisms.

Corollary 3.2.

A partially ordered set of degrees $\langle \Pi / \equiv_{In-e}; \leqslant_{In-e} \rangle$ *is not an upper half-lattice.*

Corollary 3.3.

There is a maximal degree of negative representability of linear orders with endomorphisms.

Corollary 3.4.

The structure of the degrees of negative representability of linear orders with endomorphisms is infinite.

Standard representations

We will show that the $\equiv_{In-e} \subset \equiv_{In}$ attachment is its own. To do this, consider in more detail the connection between the concepts of finitely generation, the generation of a finite set of elements of an algebra of an infinite signature and the standardness of algorithmic representations of algebras.

Definition 4.1.

An algebra is called finitely generated (locally finite) if its finitely generated finite depletion exists (accordingly, any finite depletion of it locally finite).

For finite signatures, this definition is the same as classic.

Definition 4.2.

An algebra is called a generated finite set of elements if it is generated by a finite set of elements and a set of all its operations.

Standard representations

Definition 4.2 is substantially broader than definition 4.1, since from the finitely generation follows the generation of a finite number of elements. The opposite is not true. For example, let $\mathfrak{A} = \langle \omega; f_0, f_1, \ldots \rangle$, where $\forall n, x(f_n(x) = n)$. Then the algebra $\mathfrak{A}$ is generated by any of its elements, but it is locally finite.

From the point of view of computability, it is not so important whether we apply a finite number of operations or an effective infinite set of them in the process of generating algebra, but the finiteness of the set of generating elements is fundamental.

Definition 4.3.

Algorithmic representation γ of universal algebra $\mathfrak{A}$ is called standard if it reducible to any algorithmic representation of this algebra, i.e., if ν – any algorithmic representation of algebra $\mathfrak{A}$, then for suitable computable function of h is fair $\gamma = \nu h$.

Standard representations

In other words, standard representations are those that form the smallest element relative to the reducibility of representations in a set of classes of equivalent representations (modulo the relation "be mutually reducible). Clearly, not all algebras have standard representations. For example, if $\mathfrak{A}$ is an algebra of an empty signature, then it has a continuum of minimal (relative to reducibility) classes of equivalent representations.

Proposition 4.1.

Any universal algebra of an effective signature generated by a finite number of elements has a standard algorithmic representation.

Theorem 4.1.

Over any negative equivalence there is representation such a negative linear order with endomorphisms, for which this representation is standard.

Corollary 4.1.

Any unsolvable negative equivalence is the kernel of a linear order representation with endomorphisms that does not have a positive representation.

Any equivalence is the kernel of a standard representation of a suitable algebra. However, if we are talking about linear orders and, especially, orders with endomorphisms, then the situation is radically changing. Moreover, the existence of positive equivalences was noted above, over which no linear orders are representable at all.

Standard representations

The structure of *In*-degrees contains a strictly infinitely decreasing chain of degrees $\cdots \leqslant_{In} (\eta_2) \leqslant_{In} d(\eta_1) \leqslant_{In} d(\eta_0) = d(id\ \omega)$. Taking into account the existence of infinite negative equivalence, each class of which is not computable, we have the fact of embedding in the ordered set of *In*-degrees of the ordinal type $1 + \omega^*$, where ω^* is the order of dual ω.

Open question 1.

Are there incomparable infinite In-degrees?

Proposition 4.2.

Every computable linear order with at least one limit element has an unsolvable negative representation.

Definition 4.4.

A In-degree is called splittable if it contains more than one In − e-degree.

Proposition 4.3.

If $\eta_n = \eta(\alpha_1, \dots, \alpha_n)$, where $\alpha_1, \dots, \alpha_n$ are non-intersecting noncomputable and co-enumerated sets, then the degree $d_{In}(\eta_n)$ is splittable.

From here comes the important

Corollary 4.2.

$\equiv_{In-e} \subsetneq \equiv_{In}$.

Standard representations give a powerful method for comparing the degrees of negative representability of linear orders with endomorphisms and make it possible to establish close connections between m-degrees and $In - e$-degrees.

Proposition 5.1.

$\equiv_{In-e} \subseteq \equiv_m$.

Open question 2.

Is the embedding $\equiv_{In-e} \subseteq \equiv_m$ own?

Proposition 5.2.

$\leqslant_{In-e} \subsetneq \leqslant_m$.

According to proposal 5.2, if equivalences η_0, η_1 are such that $\eta_0 \nleqslant_m \eta_1$, $d_{In-e}(\eta_0) \nleqslant_{In-e} d_{In-e}(\eta_1)$.
For $\nleqslant_{In}$-reducibility this proposition is incorrect as it shows following

Proposition 5.3.

There are such negative equivalences η_0, η_1, lying in various In-degrees that $\eta_0 \nleqslant_m \eta_1$, but $\eta_0 \leqslant_{In} \eta_1$.

It turned out that any two $In-e$-degrees comparable to relation $\leqslant_{In-e}$ lie in one In-degree.

Proposition 5.4.

If $d_{In-e}(\eta_1) \leqslant_{In-e} d_{In-e}(\eta_2)$, then $d_{In}(\eta_1) = d_{In}(\eta_2)$.

There is an important

Proposition 5.5.

$\equiv_m \subseteq \equiv_{In}$.

Recall that a subset of a partial order is called an antichain if no pair of different elements of it is comparable with respect to a given order.

Theorem 5.1.

There is a sequence of negative equivalences $\eta_0, \eta_1, \ldots,$ for which the corresponding sequence of m-degrees is strictly increasing relative to the order of $\leqslant_m$ по типу ω, sequence of In-degrees – strictly decreasing relative to $\leqslant_{In}$ by type ω^* (note here that natural embedding $\{d_m(\eta_n)\} \longmapsto \{d_{In}(\eta_n)\}$ is antiisomorphism), and the sequence of $In-e$-degrees relative to $\leqslant_{In-e}$ forms an antichain.

Proposition 5.6.

Let η_1, η_2 positive equivalences, which are the kernels of standard numberings of suitable linear orders with endomorphisms. Then fairly:
1) $d(\eta_1) \leqslant_{lp-e} d(\eta_2) \Rightarrow \eta_1 \equiv_{lp} \eta_2$;
2) $d_{lp-e}(\eta_1) = d_{lp-e}(\eta_2) \Rightarrow d_m(\eta_1) = d_m(\eta_2)$;
3) $\eta_1 \leqslant_{lp-e} \eta_2 \Leftrightarrow \eta_1 \equiv_m \eta_2$.

Now we introduce another concept of the degree of negative representability of the linear order "intermediate" between In-degrees and $In - e$-degrees. The concept of $In - e$-degree is in a sense effectively "unlimited", a very powerful tool. Moreover, we note that multi-place operations consistent with linear order can be interpreted through single-place (translations). Thus, ordered groups, rings, etc., arise.

Recall that translation is called a single-place operation in the functional signature of the system, it can be with parameters as fixed elements of the main set of the system. The oparation f from two or more arguments will be called $\leq$-vadmissible (relative to the order of $\leq$) if $\overline{x} \leq \overline{y} \Rightarrow f(\overline{x}) \leq f(\overline{y})$. For a single-seat operation, $\leq$-admissiblity means that it is a linear order endomorphism.

Proposition 5.7.

If all operations of an algebraic system in which the linear order $\leq$ is given are $\leq$-admissible, then all translations are endomorphisms. The opposite is true also, that is, if all translations are consistent with $\leq$, then all operations are $\leq$-admissible.

Thus, the classical concept of an operation consistent with an order is included in the concept of a computable family of endomorphisms of this order.

From a practical point of view, linear orders with a finite number of endomorphisms are more foreseeable.
Let η_1, η_2 be negative equivalences.

Definition 5.1.

We say that η_1 is $In - e_k$-reduced to η_2 (in the designations $\eta_1 \leqslant_{In-e_k} \eta_2$), if every linear order with no more than k ($k \in \omega$) endomorphisms, negatively represented over η_1, negatively represent over η_2.

As above, $In - e_k$-degrees will be considered in the context of the absence of degrees of finite negative equivalences.
Directly from the definition follows the property $\equiv_{In-e_{k+1}} \subseteq \equiv_{In-e_k}$. Note that $\equiv_{In-e_0} = \equiv_{In}$.

Already at the level of $k = 2$, fundamental differences arise in the structure of In degrees and $In - e_2$ degrees, which confirms

Proposition 5.8.

There are such negative equivalencies η_0, η_1, that $\eta_0 \nleqslant_{In-e_2} \eta_1$ и $\eta_1 \nleqslant_{In-e_2} \eta_0$, but $\eta_0 \leqslant_{In} \eta_1$.

Corollary 5.1.

In the set of $Ip - e_k$-degrees at $k \geqslant 2$ there are incomparable elements and there is a maximal element.

How different are the structures of In-degrees and $In - e_1$-degrees – an open question.
Similarly, we can define $Ip - e_k$-degrees.

Thank you for your attention!!!