

Finite groups and groups of finite Morley rank: more connections

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Meeting dedicated to the 75th birthday of
Professor Bruno Poizat

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Dear Bruno, is this really you?



Dedicated to Bruno Poizat

In this talk:

More results which emphasise close connections and analogies between

- ▶ ω -stable groups of finite Morley rank, and
- ▶ finite groups and algebraic groups.

Setup

V : a connected abelian group of finite Morley rank.

G : a finite set of definable isomorphisms $V \longrightarrow V$
closed under composition and inversion.

We say : the group G acts on V **definably**.

Size and complexity of finite groups

H : finite group

p : a prime number.

We introduce:

$d_p(H)$: the minimal degree of a faithful linear representation of H over the a.c. field $\overline{\mathbb{F}_p}$.

$r_p(H)$: the minimal Morley rank of a connected infinite elementary abelian p -group V of finite Morley rank such that H acts on V faithfully and definably.

The result

$d_p(H)$: the minimal degree of a faithful linear representation of H over the a.c. field $\overline{\mathbb{F}_p}$.

$r_p(H)$: the minimal Morley rank of a connected infinite elementary abelian p -group V of finite Morley rank such that H acts on V faithfully and definably.

Theorem

In this notation,

$$d_p(H) = r_p(H).$$

Corollary

Corollary via Larsen-Pink (2011). There is a function

$$J : \mathbb{N} \longrightarrow \mathbb{N}$$

with the following property:

If H is a finite simple group which acts definably and faithfully on a connected elementary abelian p -group of Morley rank $n > 0$ then either

- ▶ $|H| \leq J(n)$, or
- ▶ H is a group of Lie type in characteristic p .

Jordan type theorems, history

Camille Jordan, 1878:

There is a function

$$J : \mathbb{N} \longrightarrow \mathbb{N}$$

with the following property: every finite subgroup of GL_n over a field of characteristic zero possesses an abelian normal subgroup of index $\leq J(n)$.

Jordan type theorems, today

Guld 2020: an impressive survey of theorems of Jordan type for finite groups in algebraic (in characteristic 0) and differential geometries.

Guld 2020: For every group G of birational automorphisms of a variety over a field of characteristic zero there exists a constant $J \in \mathbb{N}$ such that every finite subgroup X of G contains a normal subgroup Y , s.t.

- ▶ Y is nilpotent of class at most two,
- ▶ $|X : Y| < J$.

There should be some model theory behind that.

Principal tool: enveloping algebras

V : a connected abelian group of finite Morley rank.

G : a finite set of definable isomorphisms $V \longrightarrow V$
closed under composition and inversion.

$E = E(G)$: the ring generated by G in $\text{End } V$, the **enveloping algebra** of G .

Observe

- ▶ $E = \{ a_1 g_1 + \cdots a_n g_n, \quad n = |G|, \quad a_i \in \mathbb{F}_p \}.$
- ▶ Elements of E act on V **definably**.

Wedderburn-Maltsev Theorem

Let R be a finite-dimensional associative algebra over a finite field \mathbb{F}_p of prime order p and J its radical. Then

- (a) $R = J + S$ where S is a semisimple algebra, $J \cap S = 0$ and S is the direct sum of matrix algebras

$$S = S_1 \oplus \cdots \oplus S_k, \quad S_i \simeq M_{d_i \times d_i}(\mathbb{F}_{p^{m_i}}), \quad i = 1, 2, \dots, k.$$

- (b) Let $Q = 1 + J$. Then Q is a normal p -subgroup in the multiplicative group R^* of R . Moreover, R^* is a semidirect product $R^* = Q \rtimes S^*$ of Q and the multiplicative group S^* of S . In particular,

$$S^* \simeq \mathrm{GL}_{d_1}(\mathbb{F}_{p^{m_1}}) \times \cdots \times \mathrm{GL}_{d_k}(\mathbb{F}_{p^{m_k}}).$$

Smoothly irreducible actions

G is **smoothly irreducible** on V if every connected definable G -invariant subgroup of V equals 0 or V .

Theorem

If a finite group G is smoothly irreducible on V then

$$E(G) \simeq M_n(F),$$

the full matrix algebra of $n \times n$ matrices over some finite field F of characteristic p .

Comments

This work is a child of lockdown. Its truly bizarre story is told in

- ▶ A. Borovik, *A view from lockdown: mathematics discovered, invented, and inherited*. arXiv:2103.04101 [math.HO].
- ▶ A shorter version in: **Math in the Time of Corona** (Alice Wonder, ed.). Springer, 2020.
https://doi.org/10.1007/16618_2020_6

Comments, continued

- ▶ The model-theoretic assumptions in the Theorem can perhaps be relaxed.
- ▶ Was proved for application in an ongoing research with
Ayşe Berkman
but a more advance result was used later.
- ▶ Motivated by research on permutation groups of finite Morley rank by
Tuna Altınel, Adrien Deloro, Josh Wiscons.
- ▶ Our project started in a paper of 2008 with **Gregory Cherlin.**

Ayşe Berkman



- ▶ Ayşe was on trial for signing *The Peace Petition*.
- ▶ About 800 defendants (12 mathematicians), 2000 court hearings.
- ▶ Ayşe's powerful speech in court on 10 January 2019 deserves to be wider known:
it is a statement of moral responsibilities of a mathematician,
<https://polytropy.com/2019/01/12/a-defense/>
- ▶ All acquitted due to a decision of the Constitutional Court.

Tuna Altinel



On arrival to Lyon,
11 June 2021:

“There are
thousands of Tuna
Altinels without a
Support
Committee.”

Linearisation of actions of simple algebraic groups.

Theorem

Let K be an algebraically closed field of characteristic $p > 0$ and G is the group of points over K of a simple algebraic group defined over \mathbb{F}_p .

Assume that G acts definably and irreducibly on an elementary abelian p -group V of finite Morley rank.

Then V has a structure of a finite dimensional K -vector space compatible with the action of G and G is a Zariski closed subgroup of $\mathrm{GL}_K(V)$.

This theorem started as Question B.38 in the book with **Ali Nesin** of 1994.

Surprisingly this was not known. . .

Corollary

G is a simple algebraic group over an algebraically closed field K of characteristic $p > 0$

V a unipotent group of exponent p

$H = V \rtimes G$ is also an algebraic group over K

V does not have closed G -invariant subgroups other than 0 and V .

Then

V has a structure of a finite dimensional vector space over K invariant under the action of G .

Idea of proof

K_0 algebraic closure of the prime field in K

$K_1 < K_2 < \dots$ finite fields, $\bigcup_{i=1}^{\infty} K_i = K_0$.

$G_i = G(K_i)$ finite groups.

$G_0 = G(K_0)$

$E_i = E(G_i)$ enveloping algebras.

$E = \bigcup_{i=1}^{\infty} E_i$ locally finite algebra.

$E = E(G_0)$

► Analysis of E yields

$E \simeq M_n(K_0)$, the full matrix algebra over K_0

► Then we show that if $Z = Z(E)^*$ then

$M = C_G(Z) = G$, that is, $[Z, G] = 1$.

Last step

Macpherson–Pillay 1995, and Deloro–Wagner 2020:

Let D be the ring of all definable endomorphisms of V and $Z = C_D(G)$. Assume that Z is infinite. Then

1. Z is an a.c. field definable in $V \rtimes G$ and the action of Z on V gives V a structure of a finite dimensional Z -vector space (with a Z -linear action of G).
2. G is a Zariski closed subgroup in $\mathrm{GL}_Z(V)$.
3. The enveloping algebra (over Z) $R = R(G)$ is the full matrix algebra $\mathrm{End}_Z(V)$ and is definable in $V \rtimes G$.

Intermediate subgroups.

Fact:

If in previous notation (G, M) is a structure of finite Morley rank with M a subgroup of G containing G_0 then $M = G$.

Proof follows from **Poizat 2001** and **Mustafin 2004**.

Bruno Poizat

This result plays crucial role in the proof:

Poizat 2001:

V : a finite dimensional vector space over an algebraically closed field K of characteristic $p > 0$

$H < \mathrm{GL}(V)$ an infinite simple group such that the structure $(\mathrm{GL}(V), H)$ has finite Morley rank.

Then H is a simple algebraic group over K .

Many thanks for your attention!