NEGATIVE REPRESENTABILITY DEGREE STRUCTURES OF LINEAR ORDERS WITH ENDOMORPHISMS

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Linear orders with computable endomorphisms

Known [1], that there is a positively representable linear order that has no computable representations. On the other hand ,[2], every negatively represented linear order has a computable representation. Therefore, the question of the existence of computable representations for negatively representable linear orders with endomorphisms is fundamental.

Theorem 1.

There is a negatively representable linear order with two endomorphisms, which does not have positive representations.

Corollary 1.

There is a negatively representable linear order with endomorphisms that does not have solvable representations.

[1] L. Feiner. Hierarchies of Boolean algebras. Journal of Symbolic Logic, **35** (2), 363–373, 1970.

[2] H.Kh. Kasymov, R.N. Dadazhanov. Negative dense linear orders. Sib. matem. journal, **58** (6), 1306-1331, 2017.

A classical example of order and endomorphism on it is a natural series with a natural order and function $x+1$.
The algebra $S=\langle\omega;s\rangle$ (without an order relation) is computably stable with respect to positive representations, i.e. any of its positive representations is computably isomorphic to the simplest.
On the other hand ,[3], there is an nonsolvable negative representation of this algebra. Against this background, the importance of order from an algorithmic point of view demonstrates the following

Proposition 1.

Any negative representation of the natural order $S_{\leq}$ of natural numbers with the function $x+1$ is solvable.

[3] B. Khoussainov, T. Slaman, P. Semukhin. $\prod_1^0$-Presentasions of Algebras. Archive for Mathematical Logic ,**45** (6), 769-781, 2006.

Degrees of linear orders with endomorphisms

We say that the linear order $\langle L; \preceq, \varepsilon_0, \varepsilon_1 \ldots \rangle$ with endomorphisms $\varepsilon_0, \varepsilon_1 \ldots$ is computably (positively, negatively) *representable over the equivalence* η on the set of natural numbers ω, if there is such a numbering of ν for L with a numbering equivalence of η, in which all endomorphisms are computable, and the sets of ν-numbers of equality and order relations are computable (positive, accordingly negative).
For negative equivalence η through $L_e(\eta)$, we denote the class of all linear orders with endomorphisms negatively represented over η and on set Π we enter the following binary relation $\leqslant_{ln-e}$:

$$\eta_1 \leqslant_{ln-e} \eta_2 \Leftrightarrow L_e(\eta_1) \subseteq L_e(\eta_2),$$

which is a preorder on the set Π and its symmetric closure $\equiv_{ln-e}$ is an equivalence by which factorization breaks the set of all negative equivalences into $\equiv_{ln-e}$-equivalence classes.

Degrees of linear orders with endomorphisms

Corollary 2.

There are incomparable degrees of negative representability of linear orders with endomorphisms.

Corollary 3.

A partially ordered set of degrees $\langle \Pi / \equiv_{ln-e}; \leqslant_{ln-e} \rangle$ *is not an upper half-lattice.*

Corollary 4.

There is a maximal degree of negative representability of linear orders with endomorphisms.

Corollary 5.

The structure of the degrees of negative representability of linear orders with endomorphisms is infinite.

Standard representations

We will show that the $\equiv_{In-e} \subset \equiv_{In}$ attachment is its own. To do this, consider in more detail the connection between the concepts of finitely generation, the generation of a finite set of elements of an algebra of an infinite signature and the standardness of algorithmic representations of algebras.

Definition 1.

An algebra is called finitely generated (locally finite) if its finitely generated finite depletion exists (accordingly, any finite depletion of it locally finite).

For finite signatures, this definition is the same as classic.

Definition 2.

An algebra is called a generated finite set of elements if it is generated by a finite set of elements and a set of all its operations.

Standard representations

Definition 2 is substantially broader than definition 1, since from the finitely generation follows the generation of a finite number of elements. The opposite is not true. For example, let $\mathfrak{A} = \langle \omega; f_0, f_1, \dots \rangle$, where $\forall n, x(f_n(x) = n)$. Then the algebra $\mathfrak{A}$ is generated by any of its elements, but it is locally finite.
From the point of view of computability, it is not so important whether we apply a finite number of operations or an effective infinite set of them in the process of generating algebra, but the finiteness of the set of generating elements is fundamental.

Definition 3.

Algorithmic representation γ of universal algebra $\mathfrak{A}$ is called standard if it reducible to any algorithmic representation of this algebra, i.e., if ν – any algorithmic representation of algebra $\mathfrak{A}$, then for suitable computable function of h is fair $\gamma = \nu h$.

Standard representations

In other words, standard representations are those that form the smallest element relative to the reducibility of representations in a set of classes of equivalent representations (modulo the relation "be mutually reducible). Clearly, not all algebras have standard representations. For example, if $\mathfrak{A}$ is an algebra of an empty signature, then it has a continuum of minimal (relative to reducibility) classes of equivalent representations.

Proposition 2.

Any universal algebra of an effective signature generated by a finite number of elements has a standard algorithmic representation.

Standard representations

Theorem 2.

Over any negative equivalence there is representation such a negative linear order with endomorphisms, for which this representation is standard.

Corollary 6.

Any unsolvable negative equivalence is the kernel of a linear order representation with endomorphisms that does not have a positive representation.

Any equivalence is the kernel of a standard representation of a suitable algebra. However, if we are talking about linear orders and, especially, orders with endomorphisms, then the situation is radically changing. Moreover, the existence of positive equivalences was noted above, over which no linear orders are representable at all.

The structure of *In*-degrees contains a strictly infinitely decreasing chain of degrees $\cdots \leqslant_{In} (\eta_2) \leqslant_{In} d(\eta_1) \leqslant_{In} d(\eta_0) = d(id\ \omega)$. Taking into account the existence of infinite negative equivalence, each class of which is not computable, we have the fact of embedding in the ordered set of *In*-degrees of the ordinal type $1 + \omega^*$, where ω^* is the order of dual ω.

Proposition 3.

Every computable linear order with at least one limit element has an unsolvable negative representation.

Definition 4.

A ln-degree is called splittable if it contains more than one ln − e-degree.

Proposition 4.

If $\eta_n = \eta(\alpha_1, \ldots, \alpha_n)$, where $\alpha_1, \ldots, \alpha_n$ are non-intersecting noncomputable and co-enumerated sets, then the degree $d_{ln}(\eta_n)$ is splittable.

From here comes the important

Corollary 7.

$\equiv_{ln-e} \subsetneq \equiv_{ln}$.

Standard representations give a powerful method for comparing the degrees of negative representability of linear orders with endomorphisms and make it possible to establish close connections between m-degrees and $ln-e$-degrees.

Proposition 5.

$\equiv_{ln-e} \subseteq \equiv_m$.

Open question 1.

Is the embedding $\equiv_{ln-e} \subseteq \equiv_m$ *own?*

Proposition 6.

$\leqslant_{ln-e} \subsetneqq \leqslant_m$.

Proposition 7.

There are such negative equivalences η_0, η_1, lying in various In-degrees that $\eta_0 \nleqslant_m \eta_1$, but $\eta_0 \leqslant_{In} \eta_1$.

It turned out that any two $In - e$-degrees comparable to relation $\leqslant_{In-e}$ lie in one *In*-degree.

Proposition 8.

If $d_{In-e}(\eta_1) \leqslant_{In-e} d_{In-e}(\eta_2)$, then $d_{In}(\eta_1) = d_{In}(\eta_2)$.

Recall that a subset of a partial order is called an antichain if no pair of different elements of it is comparable with respect to a given order.

Theorem 3.

There is a sequence of negative equivalences $\eta_0, \eta_1, \ldots,$ for which the corresponding sequence of m-degrees is strictly increasing relative to the order of $\leqslant_m$ by type ω, sequence of In-degrees – strictly decreasing relative to $\leqslant_{In}$ by type ω^* (note here that natural embedding $\{d_m(\eta_n)\} \longmapsto \{d_{In}(\eta_n)\}$ is antiisomorphism), and the sequence of $In - e$-degrees relative to $\leqslant_{In-e}$ forms an antichain.

Proposition 9.

Let η_1, η_2 positive equivalences, which are the kernels of standard numberings of suitable linear orders with endomorphisms. Then fairly:
1) $d(\eta_1) \leqslant_{lp-e} d(\eta_2) \Rightarrow \eta_1 \equiv_{lp} \eta_2$;
2) $d_{lp-e}(\eta_1) = d_{lp-e}(\eta_2) \Rightarrow d_m(\eta_1) = d_m(\eta_2)$;
3) $\eta_1 \leqslant_{lp-e} \eta_2 \Leftrightarrow \eta_1 \equiv_m \eta_2$.

Now we introduce another concept of the degree of negative representability of the linear order "intermediate" between In-degrees and $In-e$-degrees. The concept of $In-e$-degree is in a sense effectively "unlimited", a very powerful tool. Moreover, we note that multi-place operations consistent with linear order can be interpreted through single-place (translations). Thus, ordered groups, rings, etc., arise.

Recall that translation is called a single-place operation in the functional signature of the system, it can be with parameters as fixed elements of the main set of the system. The oparation f from two or more arguments will be called $\leq$-vadmissible (relative to the order of $\leq$) if $\overline{x} \leq \overline{y} \Rightarrow f(\overline{x}) \leq f(\overline{y})$. For a single-seat operation, $\leq$-admissiblity means that it is a linear order endomorphism.

Proposition 10.

If all operations of an algebraic system in which the linear order $\leq$ is given are $\leq$-admissible, then all translations are endomorphisms. The opposite is true also, that is, if all translations are consistent with $\leq$, then all operations are $\leq$-admissible.

Thus, the classical concept of an operation consistent with an order is included in the concept of a computable family of endomorphisms of this order.

Thank you for your attention!!!