CANCELLABLE ELEMENTS OF THE LATTICE OF MONOID VARIETIES

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A non-empty class of universal algebras of the same type is called a *variety* if it is closed under subalgebras, homomorphic images and direct products.

A class of universal algebras is a variety if and only if it is axiomatized by a set of identities (Birkhoff's theorem).

The collection of all varieties forms a lattice under class-theoretical inclusion.

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Let $I(x_1, x_2, \ldots, x_n)$ be a lattice identity or a lattice quasi-identity. An element a of a lattice L is an I-element if

$$(\forall x_2, x_3, \ldots, x_n \in L) \quad I(a, x_2, x_3, \ldots, x_n).$$

An element a of a lattice L is called *neutral* if

$$(\forall x, y \in L) \quad (a \vee x) \wedge (x \vee y) \wedge (y \vee a) = (a \wedge x) \vee (x \wedge y) \vee (y \wedge a).$$

$(a] = \{x \in L \mid x \leq a\}$ is the principal ideal generated by a.

$[a) = \{x \in L \mid x \geq a\}$ is the principal filter generated by a.

An element a of L is neutral if and only if $\varphi_a : x \mapsto (x \wedge a, x \vee a)$ is an embedding from L into $(a] \times [a)$.

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A neutral element is both standard and costandard.
A standard or costandard element is cancellable.

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The previous results

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$\mathbb{MON}$ denote the lattice of monoid varieties.

$\mathbf{T}$ is the trivial variety of monoids.

$\mathbf{SL}$ is the variety of all semilattice monoids.

$\mathbf{MON}$ is the variety of all monoids.

Theorem (Gusev, 2018, 2020)

For a monoid variety $\mathbf{V}$, the following are equivalent:

(i) $\mathbf{V}$ is a neutral element of the lattice $\mathbb{MON}$;

(ii) $\mathbf{V}$ is a standard element of the lattice $\mathbb{MON}$;

(iii) $\mathbf{V}$ is one of the varieties $\mathbf{T}$, $\mathbf{SL}$ or $\mathbf{MON}$.

The set of all neutral elements of the lattice of semigroup varieties is 5-element (Volkov, 2005).

The set of all standard elements of the lattice of semigroup varieties is countably infinite. (Shaprynskiĭ and Vernikov, 2010).

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$\mathbf{C} = \operatorname{var}\{x^2 \approx x^3, xy \approx yx\}$.

Theorem (Gusev, 2018)

The varieties $\mathbf{T}$, $\mathbf{SL}$, $\mathbf{C}$ and $\mathbf{MON}$ are only costandard elements of $\mathbb{MON}$.

An element of the lattice of semigroup varieties is costandard if and only if it is neutral (Vernikov, 2011).

$\mathbf{D} = \operatorname{var}\{x^2 \approx x^3, x^2y \approx xyx \approx yx^2\}$.

Theorem

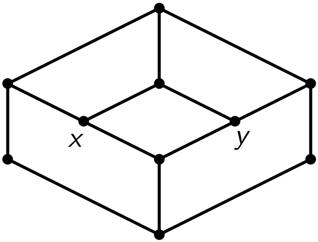
The varieties $\mathbf{T}$, $\mathbf{SL}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{MON}$ are only cancellable elements of $\mathbb{MON}$.

The set of all cancellable elements of the lattice of semigroup varieties is countably infinite (Shaprynskiĭ, Skokov and Vernikov, 2019).

Corollary

The class of all cancellable elements of $\mathbb{MON}$ forms a sublattice in $\mathbb{MON}$.

In general, the set of cancellable elements in a lattice need not form a sublattice.



An element a of a lattice L is called *modular* if

$$(\forall x, y \in L) \quad x \leq y \longrightarrow (a \vee x) \wedge y = (a \wedge y) \vee x.$$

Necessary condition of the modularity

Let V be a proper monoid variety that is a modular element of the lattice $\mathbb{MON}$. Then V satisfies the identities

$$x^2 \approx x^3 \text{ and } x^2y \approx yx^2.$$

Open problem

Classify all modular elements of the lattice $\mathbb{MON}$.

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Thank you for your attention!