Comparison of expressive capabilities of different languages of multioperations

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Multioperations

Let A be the set, $n \geqslant 0$.

- n-ary multioperation - $f : A^n \to 2^A$.
  $|A| = k$ - rank multioperation,
  n - dimension multioperation.
  M_A^n, M_A - sets of n-ary and all multioperations over A;

- multioperations $f \in M_A$, where $A = \{1, 2\}$ can be represented as mapping
  $$f : \{1, 2\}^n \to \{0, 1, 2, 3\},$$
  using the following encoding
  $$\{\varnothing\} \to 0; \{1\} \to 1; \{2\} \to 2; \{1, 2\} \to 3$$

- $f = (\alpha_1, ..., \alpha_{2^n})$ vector form n-ary f, where $\alpha_i \in \{0, 1, 2, 3\}$ and $\alpha_i = f(2^{i_1}, ..., 2^{i_n})$, but $(i_1, ..., i_n)$ is the representation of $i - 1$ base 2 n-digit number;
- n-ary empty multioperation
  $0^n(a_1, ..., a_n) = \varnothing$;
- n-ary full multioperation
  $\pi^n(a_1, ..., a_n) = A$;
- n-ary multioperation projection on i argument
  $e_i^n(a_1, ..., a_n) = a_i$;
- binary intersection multioperation
  $f_\cap(a, b) = \{a\} \cap \{b\}$;
- examples of rank 2:
  $0^3 = (00000000)$, $\pi^3 = (33333333)$
  $e_1^3 = (11112222)$, $e_2^3 = (11221122)$, $e_3^3 = (12121212)$;

Metaoperations: solvability, intersection and composition

- solvability f on to the i argument
  $(\mu_i f)(a_1, ..., a_n) = \{a | a_i \in f(a_1, ..., a_{i-1}, a, a_{i+1}, ..., a_n)\}.$

- intersection
  $(f \cap g)(a_1, ..., a_n) = f(a_1, ..., a_n) \cap g(a_1, ..., a_n).$

- composition multioperations $f \in M_A^n$ и $f_1, ..., f_n \in M_A^m$

$$f(f * f_1, ..., f_n)(a_1, ..., a_m) = \bigcup_{b_i \in f_i(a_1, ..., a_m)} f(b_1, ..., b_n)$$

Purpose of the study

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In the theory of multioperations the superassociativity identity does not hold, but only the semi-superassociativity identity is true.

Question

How different are the closed sets of simple and generalized terms for different dimensions and ranks?

Generalized and simple terms

Let $S = \{m_1, \dots, m_s\}$ be a signature, that is, it is a set of functional symbols with a fixed dimension. $e_1, \dots, e_n \in S$

Generalized term

1. if $m \in S$, then m is generalized term
2. if $t_0, t_1, \dots, t_m$ is generalized terms, then expression $(t_0 * t_1, \dots, t_m)$ is generalized term
3. if t_0 is generalized term, then $\mu_{1,\dots,n}(t_0)$ is generalized term

If $t_0 \in S$ is obligatory, then this is definition of a simple term.

Generalized and simple closure

When defining the closure of a set of multioperations using generalized terms, a generalized closure is obtained, and using simple terms, a simple closure is obtained.

Results

During the research, the following closed sets of simple and generalized terms were obtained and considered:

- Rank 2:
  1. All closed sets of unary and binary multioperations
  2. Part of closed sets of ternary multioperations
- Rank 3:
  1. All closed sets of unary multioperations
  2. Part of closed sets of binary multioperations
- Rank 4:
  1. Part of closed sets of binary multioperations

Results for ranks 3 and 4

Rank 3

Generative multioperation	Simple terms	Generalized terms
(135 320 504)	80	83
(130 326 064)	80	83
(105 026 564)	80	83

Rank 4

Generative multioperation	Simple terms	Generalized terms
(1359 3200 5040 9008)	81	84
(1300 32610 0640 01008)	81	84
(1050 0260 56412 00128)	81	84
(1009 02010 00412 910128)	81	84

Results for ranks 3 and 4

Rank 3

- $(\mu_1((\mu_1(m_1)) * (\mu_2(m_1))(\mu_1(m_1))))*$
  $(\mu_1((\mu_1(m_1)) * (\mu_2(m_1))(\mu_1(m_1))))*$
  $(\mu_2((\mu_1(m_1)) * (\mu_1(m_1))(\mu_2(m_1)))) = (711272447)$
- $(\mu_1((\mu_1(m_1)) * (\mu_2(m_1))(\mu_1(m_1))))*$
  $(\mu_2((\mu_1(m_1)) * (\mu_1(m_1))(\mu_2(m_1))))*$
  $(\mu_1((\mu_1(m_1)) * (\mu_2(m_1))(\mu_1(m_1)))) = (724174127)$
- $(\mu_1(\mu_1((\mu_1(m_1)) * (\mu_2(m_1))(\mu_1(m_1))))*$
  $(\mu_1((\mu_1(m_1)) * (\mu_2(m_1))(\mu_1(m_1))))*$
  $(\mu_2((\mu_1(m_1)) * (\mu_1(m_1))(\mu_2(m_1))))) = (135326564)$

Results for ranks 3 and 4

Rank 4

- $(\mu_1((\mu_1(m_1)) * (\mu_2(m_1))(\mu_1(m_1))))*$
  $(\mu_1((\mu_1(m_1)) * (\mu_2(m_1))(\mu_1(m_1))))*$
  $(\mu_2((\mu_1(m_1)) * (\mu_1(m_1))(\mu_2(m_1)))) = (15111215224415488815)$
- $(\mu_1((\mu_1(m_1)) * (\mu_2(m_1))(\mu_1(m_1))))*$
  $(\mu_2((\mu_1(m_1)) * (\mu_1(m_1))(\mu_2(m_1))))*$
  $(\mu_1((\mu_1(m_1)) * (\mu_2(m_1))(\mu_1(m_1)))) = (15248115481215812415)$
- $(\mu_1(\mu_1((\mu_1(m_1)) * (\mu_2(m_1))(\mu_1(m_1))))*$
  $(\mu_1((\mu_1(m_1)) * (\mu_2(m_1))(\mu_1(m_1))))*$
  $(\mu_2((\mu_1(m_1)) * (\mu_1(m_1))(\mu_2(m_1))))) = (1359326105641291012 8)$

Results

Theorem

For all ranks over 2, there is a multioperation for which the simple closure is a own subset of the generalized closure.

Hypothesis

For any set of multioperations of rank 2, its simple and generalized closures coincide.