ON RESIDUALLY NILPOTENCE OF GROUPS $F_n \rtimes_\varphi \mathbb{Z}$

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Let $\mathfrak{C}$ be some class of groups. A group G is said to be *residually* $\mathfrak{C}$*-group* or simply $\mathfrak{C}$*–residual*, if for any non-identity element $g \in G$ there exists a homomorphism φ of G to some group from $\mathfrak{C}$ such that $\varphi(g) \neq 1$. If $\mathfrak{C}$ is the class of all finite groups, then G is called *residually finite*. If $\mathfrak{C}$ is the class of finite p-groups, then G is said to be *residually p-finite*. If $\mathfrak{C}$ is the class of nilpotent groups, then G is said to be *residually nilpotent*.

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Azarov (2010) proved that any semi-direct products of finitely generated residually p-finite group by residually p-finite group is virtually residually p-finite. Next proposition follows from this result:

Any group of the form $F_n \rtimes_{\varphi} \mathbb{Z}$ is virtually residually p-finite.

Let G be a group and let $x_1, x_2, \ldots$ be elements of G. A *simple commutator* $[x_1, x_2, \ldots, x_n]$ of weight $n \geq 1$ is defined inductively by setting

$$[x_1] = x_1, \;\; [x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2 \text{ and } [x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n]$$

for $n \geq 3$.

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for $n \geq 3$.
For a group G define its transfinite lower central series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \ldots \geq \gamma_\omega(G) \geq \gamma_{\omega+1}(G) \geq \ldots,$$

where $\gamma_{\alpha+1}(G) = \langle [g_\alpha, g] \mid g_\alpha \in \gamma_\alpha(G), g \in G \rangle$ and if α is a limit ordinal, then $\gamma_\alpha(G) = \bigcap_{\beta < \alpha} \gamma_\beta(G)$.

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The smallest ordinal α such that $\gamma_\alpha(G) = \gamma_{\alpha+1}(G)$ is called the *length of the lower central series* of G.

The residually nilpotence groups of the form $\mathbb{Z}^n \rtimes_\varphi \mathbb{Z}$ were studied by Aschenbrenner and Friedl (2011). They found a criteria of residually nilpotence and residually p-finiteness for groups of the form $\mathbb{Z}^n \rtimes_\varphi \mathbb{Z}, \varphi \in \mathrm{Aut}(\mathbb{Z}^n)$. If $P_\varphi(x)$ is the characteristic polynomial of the matrix $[\varphi]$, $p_i(x) \in \mathbb{Z}[x]$, $i = 1, \dots, s$, are its non-reducible factors, then the following proposition holds.

Theorem (Aschenbrenner-Friedl, 2011)

a) $\mathbb{Z}^n \rtimes_\varphi \mathbb{Z}$ is residually nilpotent if and only if $p_i(1) \neq \pm 1$, $i = 1, \dots, s$,
b) $\mathbb{Z}^n \rtimes_\varphi \mathbb{Z}$ is residually p-finite if and only if $p_i(1) \in p\mathbb{Z}$, $i = 1, \dots, s$.

Mikhailov (2016) constructed a group G_M with one defining relation that has the lower central series of length ω^2. It answered Baumslag's question: Is the lower central series of a one-relator group of length at most $n\omega$ for some finite ordinal n?(1974)

Example (Mikhailov, 2016)

The group $G_M = \langle a, b | a^{b^2} = a(a^3)^b \rangle \cong F_2 \rtimes_\varphi \mathbb{Z}$, where $\varphi : \begin{cases} x_1 \to x_2, \\ x_2 \to x_1 x_2^3. \end{cases}$ has the lower central series of length ω^2.

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Also, the following theorem was formulated without proof for groups of the form $F_n \rtimes_\varphi \mathbb{Z}$.

Theorem (Mikhailov, 2016)

Let $G = F_n \rtimes_\varphi \mathbb{Z}$. If all groups $\overline{G}_k = (F_n^{ab})^{\otimes k} \rtimes_{\overline{\varphi}_k} \mathbb{Z}$, $k \geq 1$, are residually nilpotent, then $\gamma_{\omega^2}(G) = 1$. Wherein if G is not residually nilpotent, then the length of its lower central series is equal to ω^2.

Let us consider the semi-direct product of the free group $F_n = \langle x_1, x_2, \ldots, x_n \rangle$ and infinite cyclic group $\mathbb{Z} = \langle t \rangle$, where the conjugation by t is induced by automorphism $\varphi \in \mathrm{Aut}(F_n)$

$$F_n \rtimes_\varphi \mathbb{Z} = \langle x_1, x_2, \ldots, x_n, t \mid t^{-1} x_i t = \varphi(x_i), \quad i = 1, 2, \ldots, n \rangle.$$

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The automorphism φ induces an automorphism of the abelianization $F_n^{ab} = F_n / \gamma_2(F_n)$ that is free $\mathbb{Z}$-module. We will denote this automorphism by $\overline{\varphi}$ and its matrix by $[\overline{\varphi}]$. Denote by $\overline{G}_k = (F_n^{ab})^{\otimes k} \rtimes_{\overline{\varphi}_k} \mathbb{Z}$, $k \geq 1$, where the automorphism $\overline{\varphi}_k \in \mathrm{Aut}(F_n^{ab})^{\otimes k}$ is induced by automorphism φ. Hence, $\overline{\varphi}_1 = \overline{\varphi}$.

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Also, we will consider groups $\widehat{G}_k = \gamma_k(F_n)/\gamma_{k+1}(F_n) \rtimes_{\widehat{\varphi}_k} \mathbb{Z}$, $k \geq 1$, where $\widehat{\varphi}_k$ is the automorphism of $\mathbb{Z}$-module $\gamma_k(F_n)/\gamma_{k+1}(F_n)$ that is induced by the automorphism φ.

Proposition A

a) If $\overline{G}_k$ is residually nilpotent then $\widehat{G}_k$ is residually nilpotent.

b) If $\overline{G}_k$ is residually p-finite then $\widehat{G}_k$ is residually p-finite.

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Theorem B1

Let $G = F_n \rtimes_\varphi \mathbb{Z}$. If all groups $\widehat{G}_i = \gamma_i(F_n)/\gamma_{i+1}(F_n) \rtimes_{\widehat{\varphi}_i} \mathbb{Z}$, $i \geq 1$, are residually nilpotent, then $\gamma_{\omega^2}(G) = 1$. Wherein if G is not residually nilpotent, then the length of its lower central series is equal to ω^2.

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Theorem B2

Let $G = F_n \rtimes_\varphi \mathbb{Z}$. If all groups $\overline{G}_k = (F_n^{ab})^{\otimes k} \rtimes_{\overline{\varphi}_k} \mathbb{Z}$, $k \geq 1$, are residually nilpotent, then $\gamma_{\omega^2}(G) = 1$. Wherein if G is not residually nilpotent, then the length of its lower central series is equal to ω^2.

Theorem C

Let $G = F_n \rtimes_\varphi \mathbb{Z}$. If all groups $\widehat{G}_i = \gamma_i(F_n)/\gamma_{i+1}(F_n) \rtimes_{\widehat{\varphi}_i} \mathbb{Z}$, $i \geq 1$, are residually p-finite, then G is residually p-finite.

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Let $G = F_n \rtimes_\varphi \mathbb{Z}$. If all groups $\overline{G}_k = (F_n^{ab})^{\otimes k} \rtimes_{\overline{\varphi}_k} \mathbb{Z},\ k \geq 1$, are residually p-finite, then G is residually p-finite. In particular, the length of the lower central series of G is equal to ω.

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Theorem E

Let $G = F_n \rtimes_\varphi \mathbb{Z}$, $\varphi \in \mathrm{Aut}(F_n)$ and $[\overline{\varphi}] \in \mathrm{GL}_n(\mathbb{Z})$. If all eigenvalues of $[\overline{\varphi}]$ are integers, then G is residually nilpotent. Wherein,

a) if all the eigenvalues are equal to 1, then G is residually p-finite for any prime p;

b) if there is at least one eigenvalue that equal to -1, then G is residually 2-finite.

Theorem F

The group $G = F_2 \rtimes_\varphi \mathbb{Z}$, $\varphi \in \mathrm{Aut} F_2$ is residually nilpotent if and only if $\det[\overline{\varphi}] = 1$ and $\mathrm{tr}[\overline{\varphi}] \notin \{1, 3\}$, or $\det[\overline{\varphi}] = -1$ and $\mathrm{tr}[\overline{\varphi}] \equiv 0 \pmod 2$. At the same time, if $\det[\overline{\varphi}] = 1$, then G is residually p-finite for any prime divisor of $\mathrm{tr}[\overline{\varphi}] - 2$, and if $\det[\overline{\varphi}] = -1$, then G is residually 2-finite.

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Also, for any group of this type we find the length of its lower central series.

Theorem G

For the lower central series of $G = F_2 \rtimes_\varphi \mathbb{Z}$, $\varphi \in \mathrm{Aut} F_2$, there are only three possibilities:

a) $\gamma_\omega(G) = \gamma_2(G)$;

b) $\gamma_\omega(G) = 1$;

c) $\gamma_{\omega^2}(G) = 1$ and the length of the lower central series is equal to ω^2.

Proposition H

For the group $G = F_2 \rtimes_\varphi \mathbb{Z}$ the equalities $\gamma_\omega(G) = \gamma_2(F_2) = F_2$ hold if and only if $\det[\overline{\varphi}] = 1$, $\mathrm{tr}[\overline{\varphi}] \in \{1, 3\}$, or $\det[\overline{\varphi}] = -1$, $\mathrm{tr}[\overline{\varphi}] = \pm 1$.

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Theorem I

Let $G = F_2 \rtimes_\varphi \mathbb{Z}$. If $\det[\overline{\varphi}] = -1$ and $\mathrm{tr}[\overline{\varphi}]$ is an odd number, $\mathrm{tr}[\overline{\varphi}] \neq \pm 1$, then the length of the lower central series of G is equal to ω^2.

Proposition H

For the group $G = F_2 \rtimes_\varphi \mathbb{Z}$ the equalities $\gamma_\omega(G) = \gamma_2(F_2) = F_2$ hold if and only if $\det[\overline{\varphi}] = 1$, $\mathrm{tr}[\overline{\varphi}] \in \{1, 3\}$, or $\det[\overline{\varphi}] = -1$, $\mathrm{tr}[\overline{\varphi}] = \pm 1$.

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Lemma J

The group $G = F_2 \rtimes_\varphi \mathbb{Z}$ contains a residually nilpotent subgroup $\mathrm{gr}(x_1, x_2, t^2)$ of index 2 in all cases except $\det[\overline{\varphi}] = -1$, $\mathrm{tr}[\overline{\varphi}] = \pm 1$. In the cases $\det[\overline{\varphi}] = -1$, $\mathrm{tr}[\overline{\varphi}] = \pm 1$ it contains a residually nilpotent subgroup $\mathrm{gr}(x_1, x_2, t^4)$ of index 4. Where the group $\mathrm{gr}(x_1, x_2, t^k)$, $k = 2, 4$, is isomorphic to $F_2 \rtimes_{\varphi^k} \mathbb{Z}$

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THANK YOU!