REPRESENTATIONS of ELEMENTS of PARTIALLY COMMUTATIVE GROUPS

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1. PARTIALLY COMMUTATIVE MONOIDS.

In what follows, all graphs under consideration are undirected, without loops and multiple edges. Let $\Gamma = \langle X; E\rangle$ be a graph with the set of vertices $V(\Gamma) = X = \{x_1, \dots, x_n, \dots\}$ and the set of edges $E(\Gamma) = E$. We shall refer to Γ as a commutation or a defining graph. First partially commutative structures being studied were monoids. The notion of partially commutative monoid was introduced by Cartier and Foata in 1969 to study combinatorial problems in connection with word rearrangements.

2. (FREE) PARTIALLY COMMUTATIVE GROUPS.

A (free) partially commutative group $F(X;\Gamma)$ is defined by the presentation

$$F(X;\Gamma) = \langle X; x_i x_j = x_j x_i, \text{ if } \{x_i, x_j\} \in E\rangle.$$

The groups $F(X;\Gamma)$ were first introduced in the 1970's by Baudisch as "semifree groups" and then were studied in the 1980's by Droms calling these groups by "graph groups". Nowadays, the finitely generated groups $F(X;\Gamma)$ are usually called right-angled Artin groups.

The class of free partially commutative groups contains free, and free Abelian groups. This class is closed with respect to direct and free products. Free partially commutative groups possess a number of remarkable properties. For example, a group $F(X;\Gamma)$ is a residually torsion-free nilpotent group. Therefore free partially commutative groups are torsion-free. These groups are linear. It is observed that two free partially commutative groups $F(X;\Gamma)$ and $F(Y;\Delta)$ are isomorphic iff their commutation graphs Γ and Δ are isomorphic.
The word, conjugacy and isomorphism problems are decidable in right-angled Artin groups.

THE BLOCK DECOMPOSITION OF ELEMENTS OF FREE PARTIALLY COMMUTATIVE GROUPS

5. PARTIALLY COMMUTATIVE GROUPS. MINIMAL WORDS.

Basic properties of groups $F(X;\Gamma)$ were established by Baudisch (1977, 1981), using combinatorial methods. Call an element w in the free monoid $(X^{\pm 1})^*$ which is a minimal length representative of an element of $F(X;\Gamma)$ a minimal word. The Cancellation Lemma [Baudish, 1977] asserts that if w is a non-minimal word in $(X^{\pm 1})^*$ then w has a subword xux^{-1}, where $x \in X^{\pm 1}$, and x commutes with every letter occurring in u. This lemma implies an algorithm for finding the minimum words for elements of the group $F(X;\Gamma)$.

Let G be a group and let $g_1, g_2 \in G$. By (g_1, g_2) denotes the commutator $g_1^{-1} g_2^{-1} g_1 g_2$. Let $R = \{(x_i, x_j) \mid \{x_i, x_j\} \in E\}$. So $F(X; \Gamma)$ has presentation

$$F(X; \Gamma) = \langle X; R \rangle.$$

Then the Transformation Lemma, [Baudish, 1977] asserts that, if u and v are minimal words and $u = v$ in $F(X; \Gamma)$, then the word u may be transformed into the word v using only relations $x_i^{\varepsilon_i} x_j^{\varepsilon_j} = x_j^{\varepsilon_j} x_i^{\varepsilon_i}$, where $(x_i, x_j) \in R$, $\varepsilon_i, \varepsilon_j \in \{\pm 1\}$ (that is, without insertion or deletion of any subwords of the form $x^{\varepsilon} x^{-\varepsilon}, x \in X, \varepsilon \in \{\pm 1\}$.) It follows that, given $g \in F(X; \Gamma)$, there is a unique subset $Y \subseteq X$ such that all minimal words representing g belong to $(Y \cup Y^{-1})^*$. The set Y is called the support of g, denoted $supp(g)$. Moreover, an element g of $F(X; \Gamma)$ has a well denoted length equal to the length of a minimal word $w \in (X \cup X^{-1})^*$ representing g.

7. BLOCK (CANONICAL) DECOMPOSITIONS.

We can write elements $g \in F(X;\Gamma)$ in a canonical form, for which it is convenient to use the complement $\overline{\Gamma}$ of the defining graph Γ that is the graph with vertex set X, which has and edge $\{x_i, x_j\}$ if and only if $\{x_i, x_j\}$ is not an edge of Γ.
Given an element g of $F(X;\Gamma)$, which we regard as a minimal word, the graph $\Delta(g)$ of g is the full subgraph of $\overline{\Gamma}$ on the vertex set $supp(g)$.
If $\Delta(g)$ has components $\Delta_1, \dots, \Delta_m$ then we may write

$$g = g_1 \dots g_m, \tag{1}$$

where $\Delta(g_i) = \Delta_i$ and $(g_i, g_j) = 1$, for all i, j. We call g_i the blocks of g. As elements of $F(X;\Gamma)$, the blocks of g are uniquely determined. The notation (1) named the canonical for $g \in F(X;\Gamma)$. The canonical form is useful for solving algorithmic and algebraic questions in partially commutative groups.

PARTIALLY COMMUTATIVE METABELIAN NILPOTENT GROUPS

From now on $X = \{x_1, \ldots, x_n\}$ always stands for a finite set of vertices. Consider a variety $\mathfrak{M}$ of groups. A partially commutative group in $\mathfrak{M}$ with a commutation graph Γ is a group $F(X;\Gamma;\mathfrak{M})$ defined as

$$F(X;\Gamma;\mathfrak{M}) = \langle X; \quad x_i x_j = x_j x_i, \text{ if } \{x_i, x_j\} \in E\rangle$$

in $\mathfrak{M}$.

In particular, $F(X;\Gamma) = F(X;\Gamma;\mathfrak{G})$, where $\mathfrak{G}$ is the variety of all groups.

10. PARTIALLY COMMUTATIVE METABELIAN NILPOTENT GROUPS.

Let G be a group, $A, B \leq G$. Then $(A, B) = gp\langle (a, b)\rangle$ is the group generated by all commutators of form $(a, b), a \in A, b \in B$. So $G' = (G, G)$ is the commutant of G.

Denote by $\mathfrak{A}^2$ the variety of all metabelian groups

$$\mathfrak{A}^2 = \{G \,|\, (G', G') = 1\}.$$

Let $\mathfrak{N}_c$ be a variety of nilpotent groups of nilpotence degree at most c. This variety consists of all groups satisfying the identity $v_{c+1}(y_1, \ldots, y_{c+1}) = 1$, where $v_2 = (y_1, y_2), v_{c+1} = (v_c, y_{c+1})$.

Let $\mathfrak{N}_{2,c} = \mathfrak{A}^2 \cap \mathfrak{N}_c$.

A partially commutative metabelian nilpotent group $M_{c,\Gamma} = F(X; \Gamma; \mathfrak{N}_{2,c})$ represented as

$$M_{c,\Gamma} = \langle X;\ x_i x_j = x_j x_i,\ \text{ if } \{x_i, x_j\} \in E\rangle$$

in the variety $\mathfrak{N}_{2,c}$.

If G is a torsion-free finitely generated nilpotent group then G has a central series

$$G = G_1 \overset{a_1}{>} G_2 \overset{a_2}{>} \ldots \overset{a_s}{>} G_{s+1} = 1, \ [G_i, G] \leq G_{i+1}, \qquad (2)$$

with infinite cyclic factors. Take elements $a_1, \ldots, a_s$ such that $G_i = gp\langle a_i, G_{i+1}\rangle$.

An ordered system $\{a_1, \ldots, a_s\}$ of elements is called a Mal'tsev basis for G obtained by the central series (2).

The construction of a Mal'tsev basis of a group makes it possible to indicate a canonical form of its elements. Every element $g \in G$ can be uniquely represented in the form

$$g = a_1^{t_1} \ldots a_s^{t_s}, \ t_i \in \mathbb{Z}.$$

12. Γ_v.

Let $v(x_{i_1}, \ldots, x_{i_m})$ be a notation of an element $v \in F(X; \Gamma; \mathfrak{N}_{2,c})$ as a product of elements in $X^{\pm 1}$, where the vertices $x_{i_1}, \ldots, x_{i_m}$ occur in this notation. Let

$$\sigma(v) = \{x_{i_1}, \ldots, x_{i_m}\}.$$

Denote by Γ_v the subgraph of Γ generated by the set $\sigma(v)$ and by $\Gamma_{v,x}$ the connected component of the graph Γ_v such that this component contains a vertex $x \in \sigma(v)$.
Let us order the set X. By $\max(\Gamma_{v,x})$ denote the greatest vertex in the connected component $\Gamma_{v,x}$.

Set $(y_1, y_2, \ldots, y_m) = (\ldots(y_1, y_2), \ldots, y_m)$. Let $\mathcal{B}(M'_{c,\Gamma})$ be the set of commutators of the form

$$v = (x_{j_1}, x_{j_2}, \ldots, x_{j_m}),\ 2 \leq m \leq c,$$

in $M_{c,\Gamma}$ such that the following conditions hold:
(a) $x_{j_2} \leq x_{j_3} \leq \ldots \leq x_{j_m},\ x_{j_2} < x_{j_1}$;
(b) the vertices x_{j_1} and x_{j_2} are in different connected components of the graph Γ_v;
(c) $x_{j_1} = \max(\Gamma_{v,x_{j_1}})$.

14. THEOREM 1.

Theorem 1 [Tim., 2011, AL]. The set of elements $X \sqcup \mathcal{B}(M'_{c,\Gamma})$ is a Mal'tsev basis of the group $F(X;\Gamma;\mathfrak{N}_{2,c})$ obtained by refining the lower central series of this group.

Theorem 1 turned out to be very useful for finding a basis for a partially commutative metabelian group. The description of the basis of a partially commutative nilpotent group is completely unrelated to this theorem.

PARTIALLY COMMUTATIVE METABELIAN GROUPS

16. PARTIALLY COMMUTATIVE METABELIAN GROUPS.

A partially commutative metabelian group $M_\Gamma = F(X; \Gamma; \mathfrak{A}^2)$ represented as

$$M_\Gamma = \langle X;\ x_i x_j = x_j x_i,\ \text{ if } \{x_i, x_j\} \in E \rangle$$

in the variety $\mathfrak{A}^2$.
Among all partially commutative groups $F(X; \Gamma; \mathfrak{M})$, $\mathfrak{M} \neq \mathfrak{G},\ \mathfrak{N}_2$ the most studied case is the case of partially commutative metabelian groups. There are results obtained for centralizers and annihilators of elements of M_Γ, (Gupta., Tim.: 2009), embeddings these groups into matrix groups (Tim.: 2018), their groups of automorphisms (Tim.: 2015, 2020), values of centralizer dimensions (Tim.: 2017, 2018), direct decompositions (Romanovskii, Tim., 2020). The universal and elementary theories of these groups are investigated (Gupta., Tim.: 2009, 2011, 2012) (Tim.: 2010, 2013).

17. M'_Γ.

The group $\overline{M}_\Gamma = M_\Gamma / M'_\Gamma$ is a free Abelian group. Denote by $\overline{g}$ the image of an element $g \in M_\Gamma$ in the group $\overline{M}_\Gamma$ by natural homomorphism $M_\Gamma \to \overline{M}_\Gamma$. Then the elements $\overline{X} = \{\overline{x}_1, \ldots, \overline{x}_n\}$ form a basis of $\overline{M}_\Gamma$. Let $a_i = \overline{x}_i, \;\; A = \langle a_1, \ldots, a_n \rangle$.
The commutant M'_Γ is a right module over the group ring $\mathbb{Z}[M_\Gamma]$.
The action of an element $g \in M_\Gamma$ on $c \in M'_\Gamma$ is defined via $c^g = g^{-1}cg$.
In fact the commutant is a $\mathbb{Z}[A]-$module.
Let

$$\mathbb{Z}[A] \ni \gamma = \sum_{i=1}^{l} m_i \overline{g_i}, \quad m_i \in \mathbb{Z}.$$

Then

$$c^\gamma = (c^{m_1})^{\overline{g_1}} \ldots (c^{m_l})^{\overline{g_l}}.$$

18. A BASIS for COMMUTANT of FREE METABELIAN GROUP.

Let M be a free metabelian group with an ordered basis $X = \{x_1, \ldots, x_n\}$. By a_i denote the image of x_i by natural homomorphism $M \longrightarrow M/M'$. Then a basis of the commutant M' is the set $\mathcal{B}(M')$ consisting of all elements of the kind

$$(x_i, x_j)^{a_{i_1}^{s_{i_1}} \ldots a_{i_m}^{s_{i_m}}},$$

such that $s_{i_1}, \ldots, s_{i_m} \in \mathbb{Z},\ x_j < x_i,\ \ x_j \leq x_{i_1} < x_{i_2} < \ldots < x_{i_m}$. Let the set $\mathcal{B}(M')$ be ordered. Then each element $g \in M$ can be written uniquely in the form

$$g = x_1^{q_1} \ldots x_n^{q_n} b_1 \ldots b_m,$$

where $q_i \in \mathbb{Z},\ b_1 \leq b_2 \leq \ldots \leq b_m \in \mathcal{B}(M')$.

19. BASES of PARTIALLY COMMUTATIVE METABELIAN GROUPS.

Why elements $\mathcal{B}(M')$ do not form a basis of M'_Γ?

1. Let $\{x_i, x_j\} \in E(\Gamma)$. Then $(x_i, x_j)^{a_{i_1}^{s_{i_1}} \ldots a_{i_m}^{s_{i_m}}} = 1$ in M_Γ.
2. Let $x_i, x_{i_1}, \ldots, x_{i_m}, x_j$ be a path connecting the vertices x_i and x_j of Γ. Then

$$(x_i, x_j)^{(a_{i_1}-1)\ldots(a_{i_m}-1)} = 1$$

in M_Γ. This gives a relation over Z between elements from $\mathcal{B}(M')$.

20. BASES of PARTIALLY COMMUTATIVE METABELIAN GROUPS.

Theorem 2 [Tim., 2021, Izvestiya: Mathematics]. Let the set $X = \{x_1, \ldots, x_n\}$ be ordered. Then a basis of the commutant M'_Γ is the set we denote by $\mathcal{B}(M'_\Gamma)$ consisting of all elements v of the form

$$v = (x_i, x_j)^{a_{j_1}^{t_1} \ldots a_{j_m}^{t_m}}, \quad \{t_1, \ldots, t_m\} \subset \mathbb{Z} \setminus \{0\},\ a_i = x_i M'_\Gamma,$$

such that the following conditions are satisfied:
(1) $x_j \le x_{j_1} < x_{j_2} \ldots < x_{j_m},\ x_j < x_i$;
(2) the vertices x_i and x_j are in different connected components of the subgraph Γ_v of Γ, which is generated by all vertices of the set $\{x_i, x_j, x_{j_1}, \ldots, x_{j_m}\}$;
(3) $x_i = \max(\Gamma_{v,x_i})$, where Γ_{v,x_i} the connected component of the graph Γ_v containing x_i.

21. A CANONICAL NOTATION of ELEMENTS of PCMG.

Since M_Γ / M'_Γ is a free Abelian group, it follows a corollary.

Corollary. Let $\mathcal{B}(M'_\Gamma)$ be linearly ordered. Then any element g of the group M_Γ can be uniquely written in the form

$$g = x_1^{q_1} \dots x_n^{q_n} v_1^{r_1} \dots v_m^{r_m},$$

where $q_i, r_j \in \mathbb{Z}$ and $v_1 < \dots < v_m,\ v_j \in \mathcal{B}'(M_\Gamma)$.

PARTIALLY COMMUTATIVE NILPOTENT GROUPS AND LIE ALGEBRAS

23. PARTIALLY COMMUTATIVE NILPOTENT GROUPS.

For a group G let

$$G_{(1)} = G,\ \ G_{(c+1)} = (G_{(c)}, G).$$

The properties of partially commutative nilpotent groups

$$F(X;\Gamma;\mathfrak{N}_c) \cong F(X;\Gamma)/F_{(c+1)}(X;\Gamma)$$

are much less studied. It is known that the group $F(X;\Gamma;\mathfrak{N}_c)$ is torsion-free (Duchamp, Krob, 1992). Canonical form for elements of groups $F(X;\Gamma;\mathfrak{N}_c)$ for $c \geq 4$ was not known yet. The cases c = 2, 3 follows from theorem 1, the case c = 2 was considered by Mishchenko and Treier, 2007.

24. PARTIALLY COMMUTATIVE LIE ALGEBRAS.

The study of the (free) partially commutative Lie algebra was begun by Duchamp in 1987. Let R be a domain. A (free) partially commutative Lie $R-algebra$ $\mathcal{L}_R(X;\Gamma)$ is defined by the Lie algebra presentation

$$\mathcal{L}_R(X;\Gamma) = \langle X; [x_i, x_j] = 0, \ \text{if} \ \{x_i, x_j\} \in E\rangle,$$

where $[,]$ is the Lie brackets.
The first result on bases of free partially commutative Lie algebras $\mathcal{L}_R(X;\Gamma)$ was obtained by Duchamp and Krob 1992. But they did not give an explicit description of a basis. Using the method of Gröbner–Shirshov bases Poroshenko (2011) obtained an explicit description of bases for free partially commutative Lie algebras.

25. THE RELATION BETWEEN PCNG and PCAL.

Put $\mathcal{L}(X;\Gamma) = \mathcal{L}_{\mathbb{Z}}(X;\Gamma)$. By $\mathcal{F}$ denote a graded Lee $\mathbb{Z}-$algebra

$$\mathcal{F} = \bigoplus_{m \geq 1} F_{(m)}(X;\Gamma)/F_{(m+1)}(X;\Gamma).$$

Let us now define a family $\mathcal{A}(X)$ subsets of $\mathcal{L}(X,\Gamma)$ by induction. We set $\mathcal{A}_1(X) = X$. For $m \geq 2$, put

$$\mathcal{A}_m(X) = \{[u,v] \,|\, u \in \mathcal{A}_p(X),\ v \in \mathcal{A}_q(X),\ \ p+q=m\},$$

$$\mathcal{A}(X) = \bigcup_{m \geq 1} \mathcal{A}_m(X).$$

26. A RELATION BETWEEN PCNG and PCAL.

Let $\mathcal{L}_m(X;\Gamma)$ be a submodule of $\mathcal{L}(X;\Gamma)$ generated by $\mathcal{A}_m(X)$. Duchamp and Krob proved that there is an isomorphism α of graded Lie algebras from $\mathcal{L}(X;\Gamma)$ graded by $(\mathcal{L}_m(X;\Gamma)_{m\geq 1})$ into $\mathcal{F}$:

$$\alpha(x_i) = (x_i F_{(2)}(X;\Gamma), 1, 1, \ldots), \;\; i = 1, \ldots, n.$$

We use this result and results of Poroshenko to describe a basis of a partially commutative nilpotent group.

27. LEXICOGRAPHIC ORDERS on X^*.

The concept of basic commutators was introduced by Hall. Hall's commutators are usually used in group theory.
We will use standard commutators introduced by Chen, Fox, Lindon, 1958.
Denote by X^* the set of all words in $X = \{x_1, \ldots, x_n\}$ including the empty word denoted by 1. We also denote by $|u|$ the length of any $u \in X^*$. Let us extend an arbitrary linear order on X to a lexicographic order "$<$" on X^* as follows. Put $u < 1$ for each $1 \neq u \in X^*$ and by induction put $x_i u' < x_j v'$ if $x_i < x_j$ or $x_i = x_j, u' < v'$.

28. $\mathcal{G}(X)$.

Let

$$ALS(X) = \{u \in X^* \mid \forall u_1, u_2 \in X^*(u = u_1u_2 \longrightarrow u_2u_1 < u_1u_2)\}.$$

A word $u \in ALS(X)$ is called an associative Lyndon–Shirshov word.

Let us define a set $\mathcal{G}(X)$ of and a bar map $\mathcal{G}(X) \longrightarrow X^*$ as follows:
(a) $x_i \in \mathcal{G}(X)$ for all $x_i \in X$, $\overline{x_i} = x_i$.
(b) If $u, v \in \mathcal{G}(X)$, then $(u, v) \in \mathcal{G}(X)$ and $\overline{(u, v)} = \overline{u}\ \overline{v}$.

The bar map erases all parentheses and commas.

29. $\mathcal{G}(X)$, (X^*).

We put

$$\mathcal{G}_m(X) = \{u \mid u \in \mathcal{G}(X),\ |\overline{u}| = m\},\ \ \mathcal{G}(X) = \bigcup_{m \geq 1} \mathcal{G}_m(X).$$

Now we give a definition of the set $(X^*) \subseteq \mathcal{G}(X)$ of standard commutators:
(a) $x_i \in (X^*)$ for $i = 1, \ldots, n$.
(b) Let $z = (u, v)$. Then $z \in (X^*)$ if and only if the following conditions are true:
(b1) $\overline{z} \in ALS(X)$;
(b2) $u, v \in (X^*)$, $\overline{v} < \overline{u}$;
(b2) if $u = (u_1, u_2)$ then $\overline{u_2} \leq \overline{v}$.

30. STANDARD COMMUTATORS of CHEN, FOX, and LYNDON.

Let

$$(X^*)_m = \{u \mid u \in (X^*), |\overline{u}| = m\}.$$

If F is the free group with the basis $X = \{x_1, \dots, x_n\}$, then the set of commutators $(X^*)_m$ forms a basis of the free Abelian group $F_{(m)}/F_{(m+1)}$ for $m = 1, 2 \dots$ (see.Chen, Fox, Lyndon, 1958).

Let $u \in X^*$. By $\delta_i(u)$ denote the number of occurrences of x_i in u. Put

$$supp(u) = \{x_i \mid \delta_i(u) \neq 0\}.$$

Finally, let us define by induction a subset $\mathcal{C}(X;\Gamma)$ of (X^*) :
(a) All elements of X belong to $\mathcal{C}(X;\Gamma)$.
(b) An element $u \in (X^*)_m, m \geq 2$, belongs to $\mathcal{C}(X,\Gamma)$ if $u = (v, z)$, where v and z are elements of $\mathcal{C}(X;\Gamma)$ and there is a letter (vertex) in $supp(v)$ such that is not connected in Γ with the first letter (vertex) of z.
(c) There are no other elements in $\mathcal{C}(X;\Gamma)$.

32. $\mathcal{C}^{(m)}(X;\Gamma)$.

Let
$$\mathcal{C}_i(X;\Gamma) = \{u \in \mathcal{C}(X;\Gamma) \mid |\overline{u}| = i, \;\; i = 1, \ldots\}.$$
Let "$\prec$" be an linear order on $\mathcal{C}(X;\Gamma)$ such that $u \prec v$ if $u \in \mathcal{C}_p(X;\Gamma), \; v \in \mathcal{C}_q(X;\Gamma), \; 1 \leq p < q$.
Let
$$\mathcal{C}^{(m)}(X;\Gamma) = \bigcup_{1 \leq i \leq m} \mathcal{C}_i(X;\Gamma).$$

Theorem 3 [Tim., 2021, IJAC]. The set $\mathcal{C}^{(m)}(X;\Gamma)$ with respect to the order "$\prec$" is a Mal'tsev basis for the group $F(X;\Gamma;\mathfrak{N}_m)$ obtained by refining the lower central series.

Example 3. Let $\Gamma = \langle x_1, x_2, x_3;\ \{x_1, x_2\}\rangle$, $x_3 \prec x_2 \prec x_1$.
By construction,

$$\mathcal{C}^{(3)}(X;\Gamma) = \{x_1, x_2, x_3; (x_1, x_3), (x_2, x_3); (x_1, (x_1, x_3)),$$

$$(x_2, (x_2, x_3)), ((x_1, x_3), x_2), ((x_1, x_3), x_3), ((x_2, x_3), x_3)\}$$

is a Mal'tsev basis of the group $F(X;\Gamma;\mathfrak{N}_3)$.

PARTIALLY COMMUTATIVE METABELIAN PRO-P-GROUPS

35. PARTIALLY COMMUTATIVE METABELIAN PRO-P-GROUPS.

By a subgroup, a homomorphism, a generating set we mean a closed subgroup, a continuous homomorphism, a generating set in the topological sense, respectively.
Interesting results about right-angled Artin pro-p-groups were obtained by Snopke and P. Zalesskii (see. ArXiv, 2020).
In [Afanaseva, Tim., 2019], centralizers of elements and annihilators of commutators in partially commutative metabelian pro-p-groups were studied.

Denote by P a free metabelian pro-p-group and by P_Γ the partially commutative metabelian pro-p-group defined by a graph $\Gamma = \langle X; E \rangle$. Let $X = \{x_1, \ldots, x_n\}$. The quotient group of P_Γ by its commutant P'_Γ is a free Abelian pro-p-group A with a basis $\{a_1, \ldots, a_n\}$, where a_i is an image of x_i via the natural homomorphism $P_\Gamma \to P_\Gamma / P'_\Gamma$. This group is isomorphic to the direct sum of n copies of the additive group of the ring of integer p-adic numbers $\mathbb{Z}_p$. The action of P_Γ on P'_Γ by conjugation

$$x \to x^g = g^{-1} x g$$

defines a structure of a right module on P'_Γ over $\mathbb{Z}_p[[A]]$. The algebra is $\mathbb{Z}_p[[A]]$ identified with the power series algebra $\mathbb{Z}_p[[y_1, \ldots, y_n]]$, where $y_i = a_i - 1$.

37. A NOTATION of ELEMENTS of PCM pro-p-GROUPS.

Similarly, P' is a module over the algebra $\mathbb{Z}_p[[y_1, \ldots, y_n]]$. For this reason, any element $p \in P$ can be written in the form

$$p = x_1^{l_1} \ldots x_n^{l_n} \prod_{1 \le i < j \le n} (x_i, x_j)^{\alpha_{ij}},$$

where $l_i \in \mathbb{Z}_p, \ \ \alpha_{ij} \in \mathbb{Z}_p[[y_1, \ldots, y_n]]$.

38. BASES of PARTIALLY COMMUTATIVE METABELIAN PRO-P-GROUPS.

The following theorem is analogous to Theorem 2 and gives a description of a basis for the commutant of a partially commutative metabelian pro-p-group.

39. BASES of PARTIALLY COMMUTATIVE METABELIAN PRO-P-GROUPS

Theorem 4 [Tim., 2021, AL]. Let P_Γ be a partially commutative metabelian pro-p-group. Let the set $\{x_1, \ldots, x_n\}$ of vertices of a graph Γ be ordered. Then a basis $\mathcal{B}(P'_\Gamma)$ of the commutant P'_Γ over $\mathbb{Z}_p$ is the set of all elements v of the form

$$v = (x_i, x_j)^{y_{j_1}^{s_1} \ldots y_{jm}^{sm}}, \{s_1, \ldots, s_m\} \subset \mathbb{N},$$

such that the following conditions are satisfied:
1) $x_j \leq x_{j_1} < \ldots < x_{j_m},\ x_j < x_i$;
2) the vertices x_i, x_j are in different connected components of graph Δ_v generated by all vertices of the set $\{x_i, x_j, x_{j_1}, \ldots, x_{j_m}\}$;
3) $x_i = \max\{\Delta_{v,x_i}\}$, where Δ_{v,x_i} the connected component of the graph Δ_v containing x_i.

THANK YOU