Criterion for binarity of almost ω-categorical weakly o-minimal theories

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The 14^{th} International Summer School-Conference
"Problems Allied to Universal Algebra and Model Theory"
in occasion of 75^{th} Birthday of Professor Bruno Poizat,
Erlagol, 23-28 June 2021

Weak o-minimality

The lecture concerns the notion of *weak o-minimality* originally deeply studied in joint work by D. Marker, H.D. Macpherson and C. Steinhorn (2000).

A subset A of a linearly ordered structure M is said to be *convex* if for any $a, b \in A$ and $c \in M$ whenever $a < c < b$ we have $c \in A$.

A *weakly o-minimal structure* is a linearly ordered structure $M = \langle M, =, <, \ldots \rangle$ such that any definable (with parameters) subset of the structure M is a union of finitely many convex sets in M.

Real closed fields with a proper convex valuation ring provide an important example of weakly o-minimal (not o-minimal) structures.

Almost ω-categoricity

DEFINITION [K. Ikeda, A. Pillay, A. Tsuboi (1998), S.V. Sudoplatov (2014)]

Let T be a complete theory, $p_1(x_1), \ldots, p_n(x_n) \in S_1(\emptyset)$. A type $q(x_1, \ldots, x_n) \in S_n(\emptyset)$ is said to be a $(p_1, \ldots, p_n)$-*type* if $q(x_1, \ldots, x_n) \supseteq \bigcup\limits_{i=1}^{n} p_i(x_i)$. The set of all $(p_1, \ldots, p_n)$-types of the theory T is denoted by $S_{p_1,\ldots,p_n}(T)$. A countable theory T is said to be *almost ω-categorical* if for any 1-types $p_1(x_1), \ldots, p_n(x_n) \in S_1(\emptyset)$ there are only finitely many types $q(x_1, \ldots, x_n) \in S_{p_1,\ldots,p_n}(T)$.

Observe that almost ω-categorical weakly o-minimal theories are not small in general. As an example of such a theory we can consider the structure $M = \langle \mathbb{Q}, <, q \rangle_{q \in \mathbb{Q}}$. Then $Th(M)$ has 2^ω (continuum) 1-types over $\emptyset$, i.e. it is not small.

Disjunct union of structures

DEFINITION [R.E. Woodrow (1976)]

The *disjunct union* $\bigsqcup_{n\in\omega} \mathcal{M}_n$ of pairwise disjoint structures $\mathcal{M}_n$ with pairwise disjoint predicate signatures Σ_n, $n \in \omega$, is said to be a structure of the signature $\bigcup_{n\in\omega} \Sigma_n \cup \{P_n^{(1)} \mid n \in \omega\}$ with the universe $\bigsqcup_{n\in\omega} M_n$, $P_n = M_n$, and interpretations of predicate symbols from Σ_n coinciding with their interpretations in structures $\mathcal{M}_n$, $n \in \omega$.

Consider the disjunct union of countably many copies of the Ehrenfeucht example with three countable models ordered by the type ω. This theory has countably many non-isolated 1-types over $\emptyset$, and therefore it has maximal number of countable models. Observe that this theory is almost ω-categorical and weakly o-minimal.

The convexity rank of a set

Throughout in the lecture we consider an arbitrary complete theory T, where M is a sufficiently saturated model of T.
Below we extend the definition of convexity rank of a formula with one free variable [K., 1998] on arbitrary sets (non-necessarily definable):
Let T be a weakly o-minimal theory, M be a sufficiently saturated model of T, $A \subseteq M$.
The convexity rank of the set A ($RC(A)$) is defined as follows:
1) $RC(A) = -1$ if $A = \emptyset$.
2) $RC(A) = 0$ if A is finite and non-empty.
3) $RC(A) \geq 1$ if A is infinite.

The convexity rank of a set

4) $RC(A) \geq \alpha + 1$ if there exists a parametrically definable equivalence relation $E(x, y)$ such that there are $b_i \in A, i \in \omega$, that satisfy the following:

- For any $i, j \in \omega$, whenever $i \neq j$ we have $M \models \neg E(b_i, b_j)$.
- For every $i \in \omega \quad RC(E(M, b_i)) \geq \alpha$ and $E(M, b_i)$ is a convex subset of A.

5) $RC(A) \geq \delta$ if $RC(A) \geq \alpha$ for all $\alpha \leq \delta$ (δ is limit).
If $RC(A) = \alpha$ for some α then we say that $RC(A)$ is defined.
Otherwise (i.e. if $RC(A)) \geq \alpha$ for all α), we put $RC(A) = \infty$.

The *convexity rank of a formula* $\phi(x, \bar{a})$, where $\bar{a} \in M$, is defined as the convexity rank of the set $\phi(M, \bar{a})$, i.e.
$RC(\phi(x, \bar{a})) := RC(\phi(M, \bar{a}))$.

p-preserving convex-to-right (left) formulas

DEFINITION [B.S. Baizhanov (1996)]

Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$, $p \in S_1(A)$ be non-algebraic.

(1) An A-definable formula $F(x, y)$ is said to be *p-preserving* if there exist α, γ_1, $\gamma_2 \in p(M)$ such that $[F(M, \alpha) \setminus \{\alpha\}] \cap p(M) \neq \emptyset$ and $\gamma_1 < F(M, \alpha) \cap p(M) < \gamma_2$.

(2) A p-preserving formula $F(x, y)$ is said to be *convex-to-right (left)* if there exists $\alpha \in p(M)$ such that $F(M, \alpha) \cap p(M)$ is convex, α is a left (right) endpoint of the set $F(M, \alpha) \cap p(M)$ and $\alpha \in F(M, \alpha)$.

Equivalence-generating formulas

DEFINITION [B.S. Baizhanov, K. (2006)]

Let $F(x, y)$ be a p–preserving convex-to-right (left) formula. We say that $F(x, y)$ is *equivalence-generating* if for any $\alpha, \beta \in p(M)$ such that $M \models F(\beta, \alpha)$ the following holds:

$$M \models \forall x[x \geq \beta \to [F(x, \alpha) \leftrightarrow F(x, \beta)]]$$

$$(M \models \forall x[x \leq \beta \to [F(x, \alpha) \leftrightarrow F(x, \beta)]]).$$

EXAMPLE

Let $M = \langle \mathbb{Q}, =, <, R^2 \rangle$ be a linearly ordered structure, $\mathbb{Q}$ is the set of rational numbers, and for any $a, b \in M$ we have $M \models R(b, a) \Leftrightarrow a \leq b < a + \sqrt{2}$. Let $p(x) := \{x = x\}$. We can establish that $p(x)$ determines a complete type over $\emptyset$, $R(x, y)$ is a non-equivalence-generating p-preserving convex-to-right formula.

Properties of p-preserving convex-to-right formulas

LEMMA 1 [B.S. Baizhanov, K. (2006)]

Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$, $p \in S_1(A)$ be non-algebraic, $F(x, y)$ be a p-preserving convex-to-right formula. Suppose that $F(x, y)$ is not equivalence-generating. Then there exist α, $\beta \in p(M)$ such that

$$M \models F(\beta, \alpha) \wedge \exists x(\neg F(x, \alpha) \wedge F(x, \beta)).$$

LEMMA 2 [B.S. Baizhanov, K. (2006)]

Let T be a weakly o-minimal theory, $M \models T$, $A \subseteq M$, $p \in S_1(A)$ be non-algebraic, $F(x, y)$ be a p-preserving convex-to-right (left) formula. Then the formula $F'(x, y) := \exists z(F(z, y) \wedge F(x, z))$ is also p–preserving convex-to-right (left).

Behaviour of 2-formulas

THEOREM 1 [A.B. Altayeva, K. (2020)]

Let T be an almost ω-categorical weakly o-minimal theory, $p \in S_1(\emptyset)$ be non-algebraic. Then any p-preserving convex-to-right (left) formula is equivalence-generating.

Proof of Theorem 1. Assume the contrary: there is a p-preserving convex-to-right formula $F(x, y)$ that is not equivalence generating. Then by Lemma 1 there exist a model M of T and $\alpha, \beta \in p(M)$ such that

$$M \models F(\beta, \alpha) \wedge \exists x(\neg F(x, \alpha) \wedge F(x, \beta)).$$

Consider the following formula:

$$F_1(x, y) := \exists z(F(z, y) \wedge F(x, z)).$$

Behaviour of 2-formulas

By Lemma 2 $F_1(x,y)$ is also p-preserving convex-to-right. Consider the following formulas:

$$F_n(x,y) := \exists z(F_{n-1}(z,y) \wedge F(x,z)), \quad n \geq 2.$$

We can see that for any $\alpha \in p(M)$,

$$F_1(M,\alpha) \subset F_2(M,\alpha) \subset \ldots \subset F_n(M,\alpha) \subset \ldots.$$

Consider for every natural $n \geq 2$ the following set of formulas:

$$p_n(x,y) := p(x) \cup p(y) \cup \{F_n(x,y) \wedge \neg F_{n-1}(x,y)\}.$$

It is locally consistent. Then we obtain that the number of (p_1,p_2)-types is infinite, where $p_i(x_i) := p(x_i), i = 1,2$. The last contradicts the almost ω-categoricity of T.

Behaviour of equivalence relations

COROLLARY [A.B. Altayeva, K. (2020)]

Let T be an almost ω-categorical weakly o-minimal theory, $M \models T$, $p \in S_1(\emptyset)$ be non-algebraic, $E(x, y)$ be an $\emptyset$-definable equivalence relation partitioning $p(M)$ into convex classes, and there are at least two such classes. Then E partitions $p(M)$ into infinitely many such classes so that the induced ordering on classes is dense ordering without endpoints.

(Weak) orthogonality of families of non-algebraic 1-types

Let $p_1, p_2, \ldots, p_s \in S_1(\emptyset)$ be non-algebraic.

We say that a family of 1-types $\{p_1, \ldots, p_s\}$ is *weakly orthogonal over* $\emptyset$ if every s-tuple $\langle a_1, \ldots, a_s \rangle \in p_1(M) \times \ldots \times p_s(M)$ satisfies the same type over $\emptyset$.

We say that a family of 1-types $\{p_1, \ldots, p_s\}$ is *orthogonal over* $\emptyset$ if for every sequence $(n_1, \ldots, n_s) \in \omega^s$ and any increasing tuples $\bar{a}_1, \bar{a}'_1 \in [p_1(M)]^{n_1}$, ..., $\bar{a}_s, \bar{a}'_s \in [p_s(M)]^{n_s}$ such that $tp(\bar{a}_1/\emptyset) = tp(\bar{a}'_1/\emptyset)$, ..., $tp(\bar{a}_s/\emptyset) = tp(\bar{a}'_s/\emptyset)$, we have that $tp(\langle \bar{a}_1, \ldots, \bar{a}_s \rangle/\emptyset) = tp(\langle \bar{a}'_1, \ldots, \bar{a}'_s \rangle/\emptyset)$.

Example where weak orthogonality of a family of 1-types fails

EXAMPLE

Let $M = \langle M; <, P_1^1, P_2^1, P_3^1, f^2 \rangle$ be a linearly ordered structure such that M is a disjoint union of interpretations of unary predicates P_1, P_2 and P_3, and $P_1(M) < P_2(M) < P_3(M)$. We identify every interpretation P_i $(1 \leq i \leq 3)$ with $\mathbb{Q}$. The symbol f is interpreted by a partial binary function with $Dom(f) = P_1(M) \times P_2(M)$ and $Range(f) = P_3(M)$ and it is defined by the identity $f(a, b) = a + b$ for all $(a, b) \in \mathbb{Q} \times \mathbb{Q}$.
It can be proved that $Th(M)$ is a weakly o-minimal theory.
Obviously, $p_i(x) := \{P_i(x)\}$ determine complete 1-types over $\emptyset$ for each $1 \leq i \leq 3$. The types p_1, p_2 and p_3 are pairwise weakly orthogonal, but $\{p_1, p_2, p_3\}$ is not weakly orthogonal over $\emptyset$.

Theorem of orthogonality of a family of pairwise orthogonal 1-types

PROPOSITION 1

Let T be an almost ω–categorical weakly o-minimal theory, $M \models T$, $p_1, p_2, \ldots, p_m \in S_1(\emptyset)$ be non-algebraic pairwise weakly orthogonal 1-types. Suppose that every non-algebraic 1-type over $\emptyset$ has finite convexity rank. Then $\{p_1, p_2, \ldots, p_m\}$ is weakly orthogonal over $\emptyset$.

THEOREM 2

Let T be an almost ω–categorical weakly o-minimal theory, $M \models T$, $p_1, p_2, \ldots, p_s \in S_1(\emptyset)$ be non-algebraic pairwise weakly orthogonal 1-types. Suppose that every non-algebraic 1-type over $\emptyset$ has finite convexity rank. Then $\{p_1, p_2, \ldots, p_s\}$ is orthogonal over $\emptyset$.

Criterion for binarity

A complete theory T is said to be *binary* if any formula of the theory T is equivalent in T to a boolean combination of formulas with at most two free variables.
A. Pillay and C. Steinhorn (1986) have completely characterized all ω-categorical o-minimal theories. Their characterization implies binarity of these theories.

THEOREM 3 [K., 2007]

Let T be an ω–categorical weakly o-minimal theory. Then T is binary iff T has finite convexity rank.

THEOREM 4

Let T be an almost ω–categorical weakly o-minimal theory. Then T is binary iff every non-algebraic 1-type $p \in S_1(\emptyset)$ has finite convexity rank.