ON SOME EXPANSIONS OF DENSE ORDERS

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The number of pairwise non-isomorphic models of a theory T that have cardinality λ is denoted by $I(T, \lambda)$.

Definition 1. A theory T is called *Ehrenfeucht* if $1 < I(T, \omega) < \omega$.

Definition 2. A type $p(\overline{x}) \in S(T)$ is said to be *powerful* in a theory T if every model $\mathcal{M}$ of T realizing p also realizes every type $q \in S(T)$, i.e., $\mathcal{M} \models S(T)$.

Definition 3. Let p and q be types in $S(T)$. We say that the type p *is dominated by a type* q, or p *does not exceed* q *under the Rudin-Keisler preorder* (written $p \leq_{\mathrm{RK}} q$), if $\mathcal{M}_q \models p$, that is, $\mathcal{M}_p$ is an elementary submodel of $\mathcal{M}_q$ (written $\mathcal{M}_p \preceq \mathcal{M}_q$). Besides, we say that a model $\mathcal{M}_p$ *is dominated by a model* $\mathcal{M}_q$, or $\mathcal{M}_p$ *does not exceed* $\mathcal{M}_q$ *under the Rudin-Keisler preorder*, and write $\mathcal{M}_p \leq_{\mathrm{RK}} \mathcal{M}_q$.

DMT

A poset $(M, <)$ is a *lower semilinear order* iff every pair of elements from each set of the form $\{x \in M \mid x < a\}$ is comparable.

Let $L_{\mathrm{mt}} = \{<, \sqcap\}$, where $<$ is a binary relation symbol and $\sqcap$ is a binary function symbol. A *meet-tree* is an L_{mt}-structure M such that $(M, <)$ is a lower semilinear order where every pair of elements a, b has a greatest common lower bound, their meet $a \sqcap b$. If M is a meet-tree and $g \in M$, classes of the equivalence relation defined on $\{x \in M \mid x > g\}$ by $E(x, y) := x \sqcap y > g$ are called *open cones* above g.

DMT

Finite meet-trees form a Fraïssé class, hence have a Fraïssé limit, whose theory is complete and eliminates quantifiers. A dense meet-tree is a model of the theory DMT of the Fraïssé limit of finite meet-trees.

The theory DMT of dense meet-trees is axiomatised by saying that

1. $(M, <, \sqcap)$ is a meet-tree;
2. for every $a \in M$, the structure $(\{x \in M \mid x < a\}, <)$ is a dense linear order with no endpoints; and
3. for every $a \in M$, there are infinitely many open cones above a.

The *disjoint union* $\coprod\limits_{n\in\omega}\mathcal{M}_n$ of pairwise disjoint structures $\mathcal{M}_n$ for pairwise disjoint predicate languages Σ_n, $n\in\omega$, is structure of language $\bigcup\limits_{n\in\omega}\Sigma_n\cup\{P_n^1\mid n\in\omega\}$ with the universe $\coprod\limits_{n\in\omega}M_n$, $P_n=M_n$, and interpretations of predicate symbols in Σ_n coinciding with their interpretations in $\mathcal{M}_n, n\in\omega$. The *disjoint union of theories* T_n for pairwise disjoint languages Σ_n accordingly, $n\in\omega$, is the theory

$$\coprod_{n\in\omega}T_n\rightleftharpoons\mathrm{Th}\left(\coprod_{n\in\omega}\mathcal{M}_n\right),$$

where $\mathcal{M}_n\models T_n, n\in\omega$.

We consider some possibilities of simple complete expansions T of a theory $T_{\rm fdo}$ of a dense partial order with finitely many maximal chains and of the theory $T_{\rm dmt}$ of a dense meet-tree $\langle M, <\rangle$ [1]. Some expansions of these theories produce classical examples of Ehrenfeucht theories [2, 3], see also [4, Examples 1.1.1.3, 1.1.1.4].

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Main results

Theorem 1.

Let T be an expansion of $T_{\rm fdo}$ or $T_{\rm dmt}$ by countably many disjoint convex nonempty unary predicates P_n, $n \in \omega$. The following conditions are equivalent:

(1) T is Ehrenfeucht;

(2) T has finitely many nonisolated 1-types.

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Theorem 2.

Let T be an expansion of $T_{\rm fdo}$ or $T_{\rm dmt}$ by countably many disjoint convex nonempty unary predicates P_n, $n \in \omega$. The following conditions are equivalent:

(1) $I(T, \omega) = 2^\omega$;

(2) T has infinitely many nonisolated 1-types.

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Theorem 3.

Let T be an expansion of $T_{\rm fdo}$ or $T_{\rm dmt}$ by countably many distinct constants c_n, $n \in \omega$. The following conditions are equivalent:

(1) T is Ehrenfeucht;

(2) the set $C = \{c_n \mid n \in \omega\}$ has finitely many accumulation points in a saturated model of T;

(3) T has finitely many nonisolated 1-types.

Theorem 4.

Let T be an expansion of $T_{\rm fdo}$ or $T_{\rm dmt}$ by countably many distinct constants c_n, $n \in \omega$. The following conditions are equivalent:

(1) $I(T,\omega) = 2^\omega$;

(2) the set $C = \{c_n \mid n \in \omega\}$ has infinitely many accumulation points in a saturated model of T;

(3) T has infinitely many nonisolated 1-types.

Theorems 1–4 confirm the Vaught conjecture for special expansions of $T_{\rm fdo}$ and $T_{\rm dmt}$.

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Corollary.

Let T be an expansion of $T_{\rm fdo}$ or $T_{\rm dmt}$ by countably many disjoint convex unary predicates or by countably many constants. Then either T is Ehrenfeucht or $I(T,\omega)=2^\omega$.

For expansions of dense linear orders and their finite disjoint unions the results above hold [5, 6, 7, 8]. Using [5, 6] they can be spread for partial ordering analogues of quite o-minimal and weakly o-minimal theories admitting the description of distributions of countable models similar to [4, 7, 8].

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[5] B. Sh. Kulpeshov, S.V. Sudoplatov, Vaught's conjecture for quite o-minimal theories, *Ann. Pure and Appl. Logic*, **169**:1 (2017), 129–149.

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[7] B. Sh. Kulpeshov, S.V. Sudoplatov, Distributions of countable models of quite o-minimal Ehrenfeucht theories, *Eurasian Math. J.*, **11**:3 (2020), 66–78.

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Ehrenfeucht example

We set

$T_1 \rightleftharpoons \mathrm{Th}((\mathcal{T}; <, c_n)_{n\in\omega})$ and $T_2 \rightleftharpoons \mathrm{Th}((\mathcal{T}; <, c_n, c'_n)_{n\in\omega})$, where $<$ is an ordinary strict order on the set $\mathcal{T}$ of infinite dense branching tree forming a lower semilattice, constants c_n form a strictly increasing sequence, and constants c'_n form a strictly decreasing sequence, $c_n < c'_n, n \in \omega$.

The theory T_1 is the Ehrenfeucht's example with $I(T_1, \omega) = 3$, and the theory T_2 has six pairwise nonisomorphic countable models.

$I(T_1,\omega)=3$ $\quad$ $I(T_2,\omega)=6$

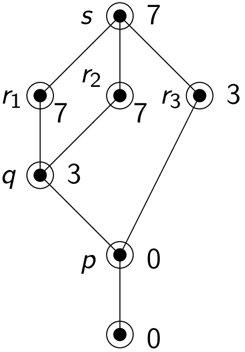
Figure 1.

Ehrenfeucht example

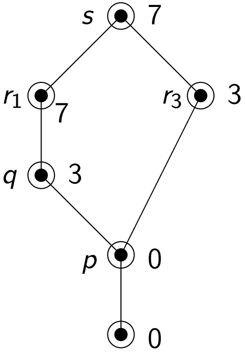
Having three sequences $(c_n)_{n\in\omega}$, $(c'_n)_{n\in\omega}$, $(c''_n)_{n\in\omega}$ of constants, where the first one strictly increases, and two others strictly decrease with respect to $<$ on the tree $\mathcal{T}$, $c_n < c'_n$, $c_n < c''_n$, $n \in \omega$, c'_i and c''_j are incomparable, $i, j \in \omega$. We get 7 prime models over tuples and 27 limit models, that is, $I(T_3,\omega)$=34.

Considering unary predicates $P_n = \{c'_n, c''_n\}$ instead of constants $c'_n, c''_n, n \in \omega$ for the corresponding theory T_4, the types r_1 and r_2 become equal. Whence the number of prime models over tuples is reduced by one, and the number of limit models is reduced by 7. Thus, $I(T_4, \omega) = 26$.

Ehrenfeucht example



$I(T_3, \omega) = 34$



$I(T_3, \omega) = 26$

Figure 2.

Note that additional expansions by strictly decreasing sequences of constants preserve the Ehrenfeuchtness of theory. In this case, the number of possibilities is defined, as above, by links between limits of sequences.

Examples above demonstrate possibilities for complications of characterizing pair (Rudin-Keisler preorder, distribution function for numbers of limit models) and quite rapid increase of number of limit models relative to constant expansions in the class of Ehrenfeucht theories.

Thank you for attention!